Mathematical Models of Water Clocks

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A Water Clock, or *clepsydra*, (a Greek word meaning water thief, [**13**]) is a device that uses the flow of water under gravity to measure time. Water flows either into, or out of, a vessel, and the height of the water in the vessel is a one-to-one function of time from the beginning of the flow.

Water clocks were important practical timekeeping devices in the ancient world. Several variations on the basic mechanism were used. Over the course of the history of water clocks, horologists—experts in the science of timekeeping—exhibited considerable ingenuity in improving their operation. In this paper, we will use tools that these innovators did not have—calculus and computers—to analyze and understand the operation of certain water clocks. We will begin with a brief outline of the types of water clocks and their history.

The simplest type of water clock, called an *outflow clepsydra*, measured time by the height of the water in a vessel that water flowed out of [see Fig. 1].

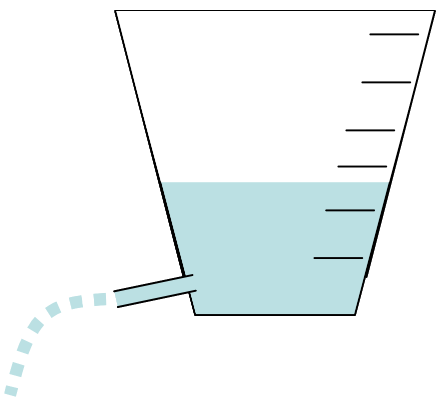

Figure 1. Simple outflow clepsydra

One of the oldest extant water clocks is an outflow clepsydra that was found in the tomb of the Egyptian pharaoh Amniotes I, who was buried around 1500 B.C.E. This discovery supports the claim of the ancient Greek historian Herodotus, who lived around 484-425 B.C.E., and who attributes the invention of the water clock to the Egyptians.

In another variation, the *inflow clepsydra*, time was measured by the height of the water in a vessel that water flowed into [Fig. 2.].

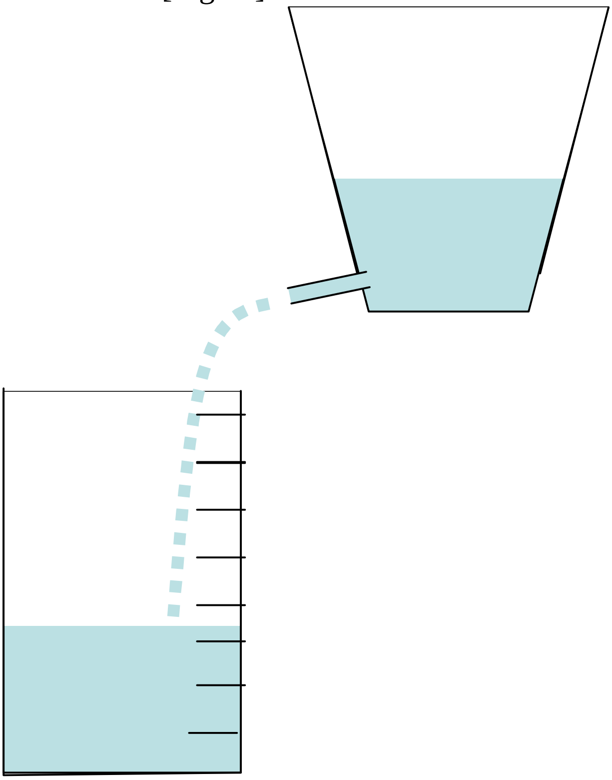

Figure 2. Simple inflow clepsydra

Inflow clepsydrae were in use in the Greek empire by the third century B.C.E. Ctesibios of Alexandria, who lived around 300-230 B.C.E., helped to develop this type of water clock, and may have invented it [**10**]. Ctesibios and other engineers of his day were the first to work on the problem that we will consider in this paper, the fact that the flow rate of water from a vessel depends on the height of the water in the vessel, so that this rate changes as the vessel drains. Their solution to the problem is shown in Fig. 3.

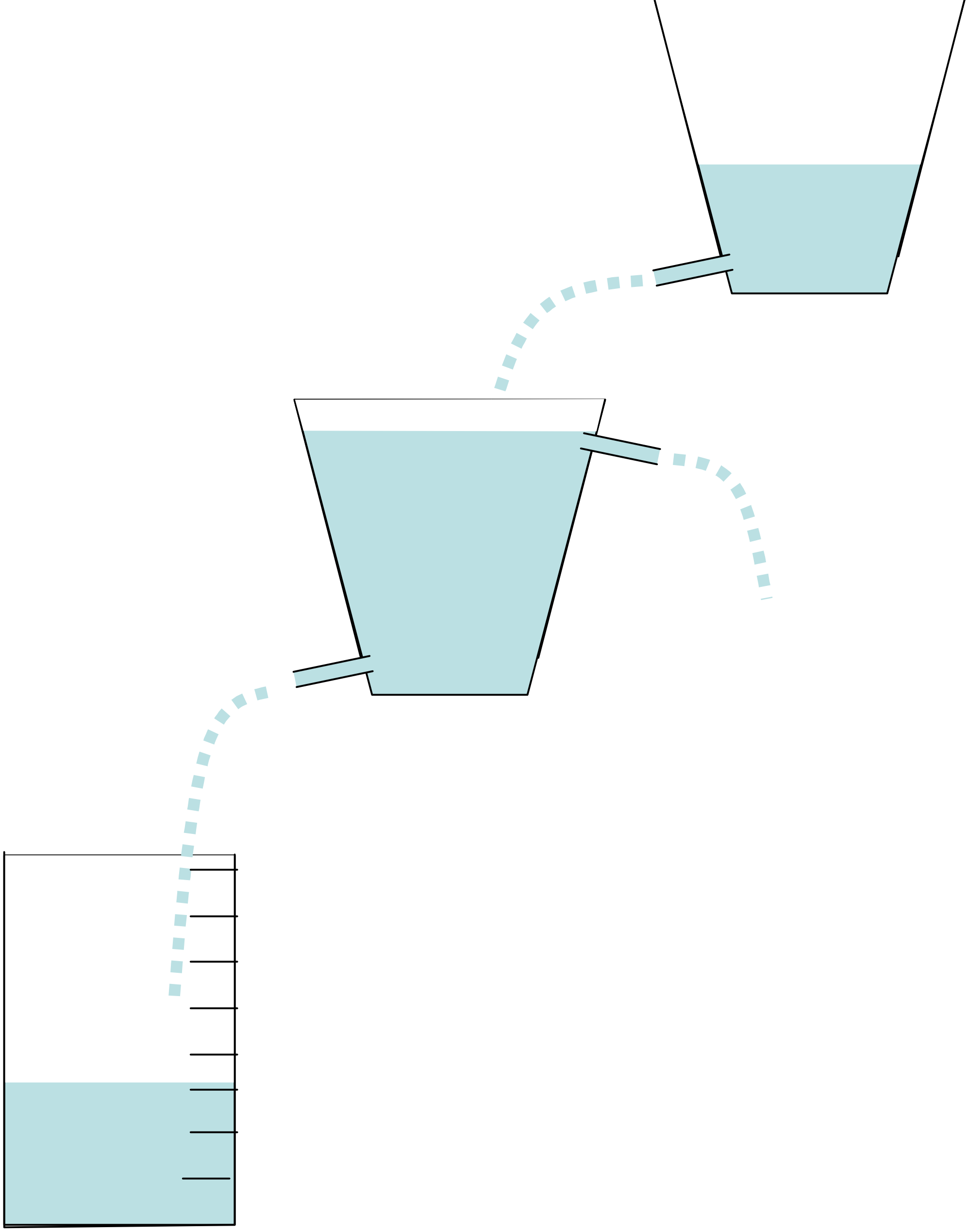

Figure 3. Inflow clepsydra with overflow tank.

The middle vessel in the configuration shown in Fig. 3 is called an *overflow tank*. It has a hole near its top to regulate the level of water; if the level of the water in that tank is constant, then the flow out the bottom of the tank is constant. A disadvantage of this scheme is that it wastes water. We shall not present a mathematical model of this configuration, but the situation is an interesting one to model, and to derive a model requires only the calculus and fluid mechanics tools that we shall discuss below. Also, this model has some interesting mathematical structure.

Another solution to the problem of variable flow rate, one that does not waste water, uses a float in the overflow tank that acts as a stopcock. It prevents the inflow of water when the level rises, and permits inflow when the water level in the overflow tank falls. Such a design is attributed to Ctesibios, who in consequence is considered the first builder of a system with feedback control [**4**].

While we are advertising modeling exercises based on clepsydrae, we should mention a variation of the inflow clepsydra, the *sinking bowl clepsydra* [Fig. 4]. In such a water clock, a bowl with a

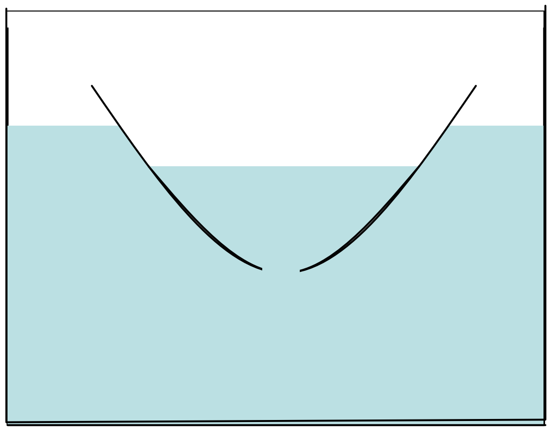

Figure 4. Sinking bowl clepsydra

hole in it is placed on the surface of water. It fills slowly and eventually sinks; the duration of its floating is taken as a unit of time. The sinking-bowl water clock seems to have been invented in India around 400 A.D. [**10**]

Inflow clepsydrae appeared in China during the Han dynasty (206 B.C.-A.D. 8), soon after the time of Ctesibios, though there is no known connection. Tan Zheng produced inflow clepsydra that survived for a millennium, and which were drawn and described in the 11th century A.D. [**10**] Chinese engineers also struggled with the problem of keeping the flow uniform. One of their solutions was the *polyvascular inflow clepsydra*, which is the type of water clock that we shall consider mathematically in this paper. One is shown schematically in Fig. 5

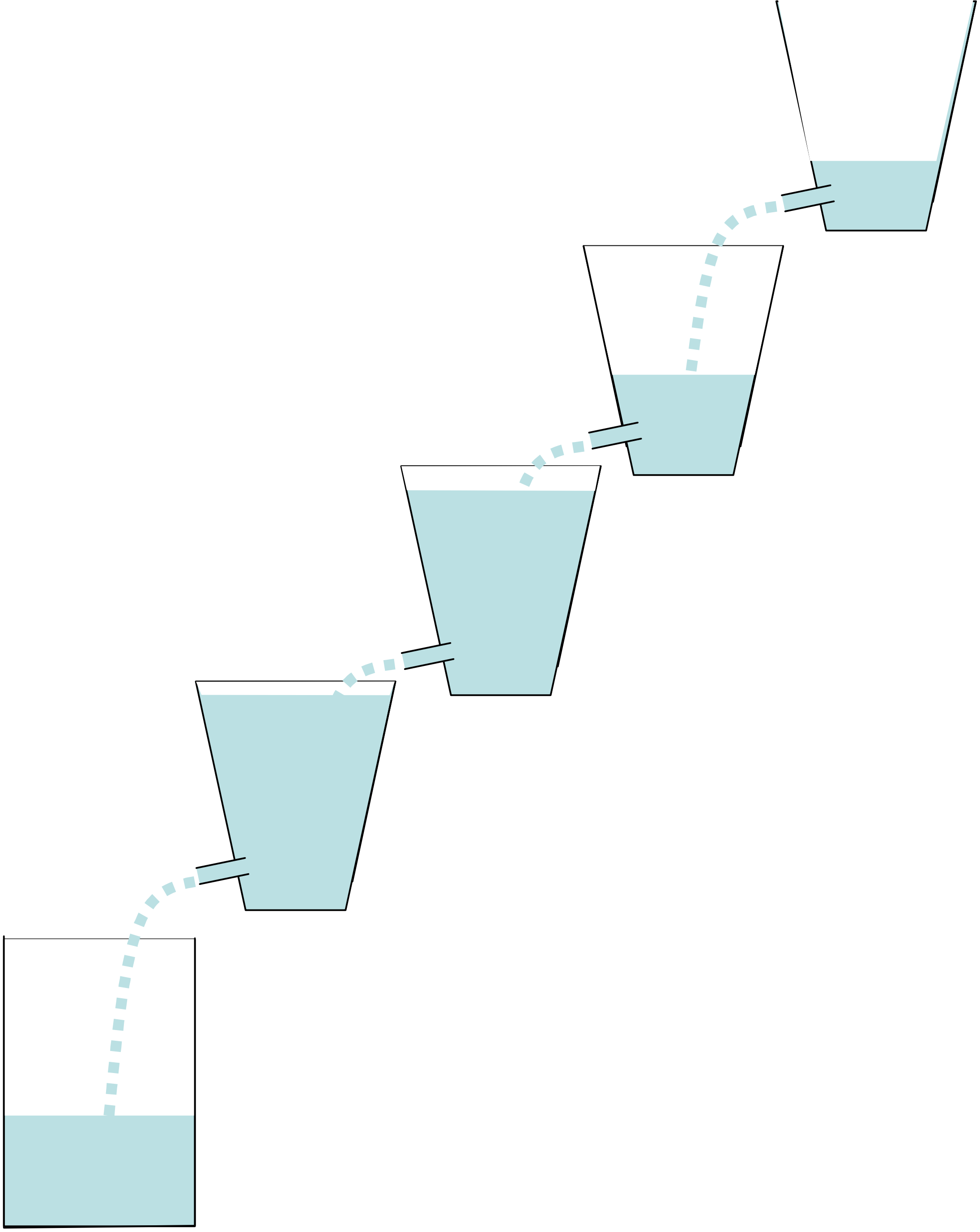

Figure 5. Polyvascular clepsydra

In a polyvascular clepsydra, a series of vessels drain successively into one another. The modeling exercise we undertook was to describe such clepsydra mathematically, and to use the mathematical model to understand how and why the work; that is, how they produce a constant flow rate from the final vessel.

Lest the reader think that the use of water clocks ended in the first millennium, we note that Galileo used a water clock in his studies of mechanics. According to MacLachlan [**3**], Galileo "measured time by weighing water that flowed out of a narrow tube at the bottom of a bucket…Although primitive in structure, this is a quite accurate clock."

The mathematical model

We take the polyvascular clepsydra to consist of N vessels, each of which is a right circular cylinder of height 1. We also take them all to be identical, and to be full initially. They drain through holes or nozzles in their bases. Vessel 1 drains into vessel 2, which drains into vessel 3, and so on. For the sake of having a complete story, we can think of there being an N+1st vessel into which the Nth vessel drains; the depth of water in this N+1st vessel, which is initially empty, is used to measure time. The goal is to have the water rise in this vessel at a constant rate.

We shall let $y_j(t)$ be the height of the water in the jth vessel at time t.

The outflow rate from a vessel will be a function of the pressure at the bottom of that vessel, that is, the pressure at the point where the water leaves the vessel. More specifically, since it is atmospheric pressure that is resisting the flow of the water out of the vessel, the outflow rate will be a function of the increase in pressure over atmospheric pressure. To simplify notation, we take atmospheric pressure to be 0. Then the pressure in the vessel is *hydrostatic* pressure [**11**, chapter 40]. This means that the pressure at a depth h below the surface of the water is $r gh$, where r is the density of water and g is the acceleration of gravity; the pressure at any depth is simply the force per unit area that is required to carry the weight the fluid above that depth.

The last, and most complex, modeling issue is whether the water experiences significant viscous drag as it flows out of the vessels. We will treat both of these cases: the case in which viscous drag is negligible, and the case in which it is dominant.

Viscosity will dominate the outflow rate if the water flows out through a nozzle that is sufficiently long and thin. A precise mathematical definition of the requisite dimensions can be made in terms of the *Reynolds number* [**11**, chapter 41]. But for our purposes a simple physical explanation should suffice. If the nozzle is sufficiently short, water that enters the nozzle with some momentum will not have much of its momentum dissipated by viscous drag during its short trip through the nozzle. Viscous drag is caused by the water's sticking to the walls of the nozzle, and if there is not a long wall, there is not much viscous dissipation. If the nozzle is so long that *all* of the entering water's momentum is dissipated by viscous drag, then we are in a situation in which viscosity dominates the flow. The diameter of the nozzle will also affect the dissipation of the water's momentum; since the dissipation is caused by the water's sticking to the wall of the nozzle, there will be more dissipation in a thin nozzle because the surface area to volume ratio—the amount of wall per unit water—will be larger.

In cases in which viscosity dominates, the outflow rate is related to the pressure at the bottom of the vessel by Poisseille's Law,

$$Q = \frac{p r^4}{8 m L} P$$

Here, Q is the outflow rate in units of volume per unit time, P is the pressure, m is the viscosity, which has units of mass per unit length per unit time, and r and L are the radius and the length of the nozzle, respectively. If we substitute the expression for the hydrostatic pressure for P in this formula, we find that in the case in which viscosity dominates the outflow, the outflow rate from the jth vessel

$$Q = \frac{p r^4 r g}{8 m L} y_j(t) \,. \tag{1}$$

If the outflow nozzle is sufficiently wide and short—for example, if it is just a hole—then viscosity should be negligible. In this case, the outflow rate is given by Torricelli's Law [**9**], which is based on Bernoulli's Law. When viscosity dominates the flow, the fluid never builds up significant momentum. But when viscosity is negligible, pressure differences in the fluid can accelerate the fluid significantly. The work done by gravity, and by the pressure gradient, in accelerating the water turns into kinetic energy. This conservation of energy is expressed by Bernoulli's Law [**11**, chapt. 40]:

$$\frac{r}{2} v^2 + P + r g (y - y_j) = 0$$

Here, v is the speed of the water and y is the distance from the bottom in the jth vessel.

At the bottom of the vessel, at the outflow nozzle, we have $P = 0$ because the pressure is the atmospheric pressure, and $y = 0$, so Bernoulli's Law gives us

$$v^2 = 2 g y_j$$

or

$$v = \sqrt{2 g y_j}$$

This is the magnitude of the velocity, which may not be directed straight out of the nozzle; this will affect the net outflow rate, which will also depend on the area of the nozzle. We would have to do some more detailed physics to account for these effects precisely, but in any case we will have

$$Q = c \sqrt{y_j} \tag{2}$$

where c is a positive constant. With Eqs. (1) and (2) in hand we are ready to write down the equations of our model. The basic law is the conservation of mass: the rate at which the volume of water in the vessel decreases is equal to the rate at which water flows out less the rate at which it flows into the vessel. No water flows into the first vessel, so the equation, in the viscosity-dominated case, is

$$A\frac{dy_1(t)}{dt} = -\frac{p\,r^4\,r\,g}{8mL}\,y_1(t)$$

where A is the cross-sectional area of the vessel. We re-scale the time so that this equation becomes

$$\frac{dy_1(t)}{dt} = -y_1(t)$$

Water flows both into and out of all of the other vessels, apart from vessel (N+1). The rate at which water flows into the j^{th} vessel is exactly the rate at which it flows out of the j-1^{st} vessel, so we have

$$\frac{dy_j(t)}{dt} = y_{j-1}(t) - y_j(t)$$

for $2 \le j \le N$. Since we assume that all of the vessels are initially full, and that they all have unit height and unit volume, our full set of equations is:

$$\begin{aligned} \frac{dy_1(t)}{dt} &= -y_1(t) \quad y_1(0)=1 \\ \frac{dy_j(t)}{dt} &= y_{j-1}(t) - y_j(t) \quad y_j(0)=1 \quad 2 \le j \le N \end{aligned} \tag{3}$$

Viscosity-Dominated Case

By using the same reasoning along with Eq. (2), we obtain the equations for the inviscid case, (here again, we have re-scaled time in order to eliminate an unimportant constant),

$$\begin{aligned} \frac{dy_1(t)}{dt} &= -\sqrt{y_1(t)} \quad y_1(0)=1 \\ \frac{dy_j(t)}{dt} &= \sqrt{y_{j-1}(t)} - \sqrt{y_j(t)} \quad y_j(0)=1 \quad 2 \le j \le N \end{aligned} \tag{4}$$

Invsicid Case

Analysis of the viscosity-dominated case

The viscosity-dominated case, which is described by Eqs. (3), provides a nice exercise in the use of exponential functions and in the application of the method of variation of parameters [**9, 12**]. What is more interesting is that the solution, interpreted in the context of our water clock problem, provides insight into the nature of Taylor polynomials [**9**] in general, and into the structure of Taylor approximations of exponential functions in particular.

The first equation in Eqs. (3) can be solved simply; its solution is the exponential function

$$y_1(t) = e^{-t}$$

The next equation, the equation for $y_2(t)$, can then be regarded as an inhomogeneous equation whose inhomogeneous term is the known function $y_1(t)$

$$\frac{dy_2(t)}{dt} = -y_2(t) + y_1(t)$$

This equation can be solved by variation of parameters. In this method, one first finds the general solution to the homogeneous equation, which in this case is $\frac{dy_2(t)}{dt} = -y_2(t)$, whose general solution is $y_2(t) = s\, e^{-t}$, where s is an arbitrary constant. One then treats the constant as a function of t, $y_2(t) = s\,(t)e^{-t}$, and plugs this expression into the inhomogeneous equation. In this case, we obtain

$$\frac{ds\,(t)e^{-t}}{dt} = -s\,(t)e^{-t} + y_1(t)$$

or, by applying the product rule,

$$\frac{ds\,(t)e^{-t}}{dt} = -s\,(t)e^{-t} + \frac{ds\,(t)}{dt}e^{-t} = -s\,(t)e^{-t} + y_1(t)$$

When we cancel the term $-s\,(t)e^{-t}$, this yields

$$\frac{ds\,(t)}{dt} = y_1(t)e^{t}$$

or

$$s\,(t) = 1 + \int_0^t y_1(s)e^{s}ds = 1 + \int_0^t e^{-s}e^{s}ds = 1 + \int_0^t ds = 1 + t$$

Here, we have used the initial condition $y_2(0)=1$ to determine that $s(0)=1$. So we have

$$y_2(t)=s(t)e^{-t}=(1+t)e^{-t}$$

We can repeat this procedure for $j=3,4,\ldots$. We find that

$$e^t y_j(t)=1+\int_0^t y_{j-1}(s)e^s ds$$

This proves, by induction, that

$$y_j(t)=(1+t+\frac{t^2}{2!}+\ldots\frac{t^{j-1}}{j-1!})e^{-t} \tag{5}$$

Thus, we have an explicit solution of the system of equations (3).

The formula in Eq. (5) shows $y_j(t)$ to be the product of e^{-t} and the j$^{\text{th}}$ Taylor polynomial of e^t, $1+t+\frac{t^2}{2!}+\ldots\frac{t^{j-1}}{j-1!}$. This suggests an approach to analyzing this solution. We can express

$$y_N(t)=(1+t+\frac{t^2}{2!}+\ldots\frac{t^{N-1}}{N-1!})e^{-t}=(e^t-\sum_{k=N}^{\infty}\frac{t^k}{k!})e^{-t}=1-e^{-t}\sum_{k=N}^{\infty}\frac{t^k}{k!}$$

Now, the original question that we set out to address was how nearly constant the flow from the N$^{\text{th}}$ vessel is, and for how long; making this flow rate constant was the purpose of the polyvascular clepsydra. The flow rate out of the N$^{\text{th}}$ vessel will be constant if $y_N(t)$ is constant. That is, since $y_N(t)$ is initially equal to 1, the flow rate will be nearly constant as long as $y_N(t)$ is approximately equal to 1, that is, as long as $e^{-t}\sum_{k=N}^{\infty}\frac{t^k}{k!}$ is small.

The expression $\sum_{k=N}^{\infty}\frac{t^k}{k!}$ is the error associated with the N$^{\text{th}}$ order Taylor approximation of e^t, so we know by Taylor's theorem [**9**] that $0\le\sum_{k=N}^{\infty}\frac{t^k}{k!}\le\frac{e^t t^N}{N!}$. (This is actually a crude estimate, in this case, unless N and t are small.) This means that $y_N(t)\ge 1-\frac{t^N}{N!}$. The following lemma gives us an estimate of how long $y_N(t)$ remains close to 1. This lemma will also provide a crucial estimate in the inviscid case.

LEMMA 1. If $\frac{t^N}{N!} = c > 0$, then $t > \frac{Nc^{\frac{1}{N}}}{e}$.

Proof. If $\frac{t^N}{N!} = c > 0$ then a little algebra yields $t = c^{\frac{1}{N}}(N!)^{\frac{1}{N}}$, so we must show that $(N!)^{\frac{1}{N}} > \frac{N}{e}$. For each value of i from 1 through N, place a rectangle of height $\log(i)$ and width 1 between $x = i-1$ and $x = i$. The sum of the areas of these rectangles is an upper Riemann sum for the integral $\int_1^N \log(t)dt$:

$$\log(N!) = \sum_{i=1}^{N} \log(i) \geq \int_1^N \log(t)dt = N\log(N) - N + 1$$

Thus

$$\frac{1}{N}\log(N!) \geq \log(N) - 1 + \frac{1}{N}$$

By exponentiating, we obtain

$$(N!)^{\frac{1}{N}} \geq e^{\log(N)-1+\frac{1}{N}} = \frac{Ne^{\frac{1}{N}}}{e} > \frac{N}{e}$$

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Suppose, for example, that we want to know when $y_N(t) = .9$. Since this occurs after the time at which $\frac{t^N}{N!} = .1$, LEMMA 1 tells us that it occurs after time $t = \frac{N(.1)^{\frac{1}{N}}}{e}$.

This crude estimate tells us that our model has, at least, a basic qualitative feature that we expect from our water clock: the drainage time from an N-vessel clock is essentially linear in N. We can obtain a better estimate by using a more precise approximation for the error term, $e^{-t}\sum_{k=N}^{\infty}\frac{t^k}{k!}$, in the Taylor approximation. We leave this as an exercise for interested readers.

Analysis of the Inviscid Case

The first equation in the inviscid model, $\frac{dy_1(t)}{dt} = -\sqrt{y_1(t)}$, with initial condition $y_1(0) = 1$, can be solved in closed form by direct integration to obtain $y_1(t) = (1 - \frac{t}{2})^2$. However, since the system of equations (4) is not linear, the variation of parameters method does not work to get us solutions of the rest of the equations as it did in the viscosity-dominated case. As best we can tell, it is impossible to integrate the equations for $y_j(t),\ j > 1$, in closed form. We must resort to estimates and numerical solutions.

Note that solutions of the inviscid model actually reach 0; the vessels drain completely in finite time. The inviscid model differs in this feature from the viscosity-dominated model, in which the height of water in each vessel approached zero only asymptotically—the vessels never drained completely. This is, to put it perhaps a bit tersely, because the improper integral $\int_0^1 \frac{dy_1}{\sqrt{y_1}}$, which gives the time at which the first vessel drains in the invsicid model, converges, whereas the analogous integral for the viscosity-dominated model, $\int_0^1 \frac{dy_1}{y_1}$, diverges.

Because $y_1(t) \equiv 0$ for $t \geq 2$, it follows that $y_2(t)$ has the form $y_2(t) = (\frac{A}{2} - \frac{t}{2})^2$ for some constant $A > 2$, from time $t = 2$ until time $t = A$, after which time $y_2(t) \equiv 0$. And, so on, for $j = 3, 4, \ldots$. This is an interesting fact, but we have not found it useful.

We start with the results of our numerical solution of Eqs. (4). These are shown in Fig. 6.

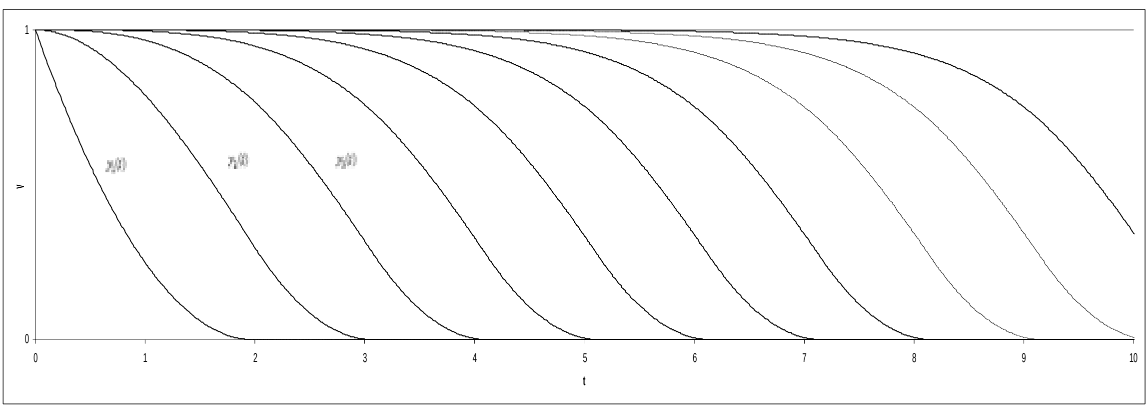

Figure 6. Numerical solution of Eqs. (4)

We begin our analysis with another lemma:

LEMMA 2: For all $j \geq 1$, there is a time $t_j > 0$ such that $y_j(t) > 0$ for $0 \leq t < t_j$ and $y_j(t) \equiv 0$ for $t \geq t_j$, and $t_{j+1} > t_j$. For all $j \geq 1$, and for all $t > 0$ $y_{j+1}(t) \geq y_j(t)$, and the inequality is strict if $t \leq t_j$.

Proof: If follows from Eq. (4) by direct integration that $y_1(t) = \begin{cases} (1-\frac{t}{2})^2 & 0 \leq t < 2 \\ 0 & 2 < t \end{cases}$, so $t_1 = 2$. It also follows from Eq. (4) that $\frac{d(y_2 - y_1)}{dt} = (\sqrt{y_1} - \sqrt{y_2}) + \sqrt{y_1}$. Since $y_1(0) = y_2(0) = 1$ and $y_1(t) = (1-\frac{t}{2})^2$ for $0 \leq t \leq 2$, this means that $\frac{d(y_2 - y_1)}{dt} > 0$ for sufficiently small times. But if, at any later time, $y_2(t) = y_1(t)$, then $\frac{d(y_2 - y_1)}{dt} = (\sqrt{y_1} - \sqrt{y_2}) + \sqrt{y_1} = \sqrt{y_1}$. This implies that $y_2(t) \geq y_1(t)$ for all $t \geq 0$ and $y_2(t) > y_1(t)$ for $t \leq t_1 = 2$. For $t > t_1$, $\frac{dy_2}{dt} = -\sqrt{y_2}$ and so by direct integration

$$y_2(t) = \begin{cases} \left((y_2(t_1))^2 - \left(\frac{t - t_1}{2} \right) \right)^2 & t_1 \leq t < t_1 + 2(y_2(t_1))^2 \\ 0 & t_1 + 2(y_2(t_1))^2 < t \end{cases}, \text{ so } t_2 = t_1 + 2(y_2(t_1))^2.$$

In the general case, for $j \geq 2$,

$$\frac{d(y_{j+1} - y_j)}{dt} = (\sqrt{y_j} - \sqrt{y_{j+1}}) + (\sqrt{y_j} - \sqrt{y_{j-1}}) = \left(\frac{y_j - y_{j+1}}{\sqrt{y_j} + \sqrt{y_{j+1}}} \right) + (\sqrt{y_j} - \sqrt{y_{j-1}}).$$ So, by the variation of parameters formula [**9**],

$$y_{j+1} - y_j = e^{\int_0^t \frac{ds}{\sqrt{y_j(s)} + \sqrt{y_{j+1}(s)}}} \int_0^t e^{-\int_0^t \frac{ds}{\sqrt{y_j(s)} + \sqrt{y_{j+1}(s)}}} \left(\sqrt{y_j(t)} - \sqrt{y_{j-1}(t)} \right) dt.$$ Thus, if $y_j > y_{j-1}$ for $0 < t \leq t_j$, then $y_{j+1} > y_j$ for $0 < t \leq t_j$. For $t > t_j$, $\frac{dy_{j+1}}{dt} = -\sqrt{y_{J+1}}$, so

$$y_{j+1}(t) = \begin{cases} \left((y_{j+1}(t_j))^2 - \left(\frac{t - t_j}{2} \right) \right)^2 & t_j \leq t < t_j + 2(y_{j+1}(t_j))^2 \\ 0 & t_j + 2(y_{j+1}(t_j))^2 < t \end{cases} \text{ so } t_{j+1} = t_j + 2(y_{j+1}(t_j))^2.$$

The result now follows by induction.

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COROLLARY. For all $j \geq 1$, $y_j(t)$ is a strictly decreasing function for $0 < t < t_j$.

Proof: This now follows directly from Eq. (4): $\frac{dy_j(t)}{dt} = \sqrt{y_{j-1}(t)} - \sqrt{y_j(t)} < 0$ for $0 < t < t_j$ by Lemma 2.

A Simple Estimate

There is a simple estimate that has the form we are seeking: By adding the first j equations we obtain $(y_1 + y_2 + ... y_j)' = -\sqrt{y_j} \geq -\sqrt{y_1 + y_2 + ... y_j}$. By integrating both sides of the inequality $\frac{(y_1 + y_2 + ... y_j)'}{\sqrt{y_1 + y_2 + ... y_j}} \geq -1$ we find that $(y_1 + y_2 + ... y_j) \geq (\sqrt{j} - \frac{t}{2})^2$. This means that $y_j(t) \geq 0$ for $t < 2\sqrt{j}$.

This does show that we can make the drainage time for the polyvascular clepsydra as long as we like by using a sufficiently large number of vessels. But our numerical results suggest that $y_j(t) \geq 0$ until $t \sim j+1$, so the estimate $2\sqrt{j}$ is a very crude lower bound for large values of j.

A Better Estimate: The Right Order in n

We can get a much better estimate with just a little bit of hard analysis. We let

$$y_j(t) = (1 - q_j(t))^2$$

(This equation constitutes a definition of $q_j(t)$.) So, for example, $q_1(t) = \frac{t}{2}$. If we expand each $y_j(t)$ in a power series near $t = 0$, it follows that

$$y_j(t) = 1 - \frac{t^j}{2^j j!} + O(t^{j+1})$$

This fact suggested the content of the following theorem.

THEOREM. *For all* j, $y_j(t) \geq (1 - \frac{t^j}{2^j j!})^2$ *for* $0 \leq t \leq 2(j!)^{\frac{1}{j}}$.

We will prove this theorem below. First, we state its important corollary, which is the result in which we are interested: it says that the time at which the j$^{\text{th}}$ vessel becomes empty is linear in j.

COROLLARY. *The function* $y_j(t) > 0$ *for* $0 \le t \le \frac{2j}{e}$. *If* $0 < \Gamma < 1$ *and* $y_j(t) \le \Gamma$, *then* $t \ge \frac{2(1-\sqrt{\Gamma})^{\frac{1}{j}}}{e} j$.

Proof. The fact that $y_j(t) > 0$ for $0 \le t \le \frac{2j}{e}$ follows directly from the theorem. The fact that $t \ge \frac{2(1-\sqrt{\Gamma})^{\frac{1}{j}}}{e} j$ if $y_j(t) \le \Gamma$ follows from the theorem and LEMMA 1 by replacing c in the statement of that lemma with $2^j(1-\sqrt{\Gamma})$.

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This estimate shows the time at which the jth vessel drains to be linear in j, which is what we expect. We can see from our numerical results that the factor of $\frac{2}{e} = .736\ldots$ is pessimistic; we should be able to replace it with a factor of 1. The question of how to derive an estimate with such a factor remains open.

We shall now prove the theorem. For simplicity, we shall first prove another lemma.

LEMMA 3. *If* $0 \le a \le 1$, *and* $0 \le q < a$ *then* $0 \le \frac{a-q}{1-q} \le a$.

Proof. It follows directly from the premises of the Lemma that $0 \le \frac{a-q}{1-q}$, since both the numerator and denominator of this expression are positive. The other inequality follows from $\frac{a-q}{1-q} = \frac{(a-aq)+(aq-q)}{1-q} = a + q\frac{a-1}{1-q} = a - q\frac{1-a}{1-q} \le a$

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Proof of the Theorem.

In terms of $q_j(t)$ we can rephrase the conclusion of the theorem as that $q_j(t) \le \frac{t^j}{2^j j!}$ for all j. We prove this by induction. We know that $q_1(t) = \frac{t^1}{2^1 1!} = \frac{t}{2}$. In terms of $q_j(t)$ the system of ODEs becomes

$$\frac{dq_j}{dt} = \frac{q_{j-1} - q_j}{2(1-q_j)}$$

and the associated initial conditions are $q_j(0) = 0 \ \forall j$ .We know by LEMMA 2 that $q_{j-1} \le q_j$. It thus follows from Lemma 3 that $\frac{dq_j}{dt} = \frac{q_{j-1} - q_j}{2(1-q_j)} \le \frac{q_{j-1}}{2}$. This means, by the induction hypothesis, that $\frac{dq_j}{dt} \le \frac{t^{j-1}}{2^j (j-1)!}$. By integration and application of the initial condition it follows that $q_j(t) \le \frac{t^j}{2^j j!}$, and the general result follows by induction.

■

A Conjecture based on Numerical Results

In the previous section we showed that the jth vessel does not drain until a time later than $\frac{2j}{e}$ and we noted that this estimate seems pessimistic in light of the numerical results that we presented in Fig. 6. Those results, which we obtained with a simple Euler integration and a very small step size, suggest that the jth vessel does not drain until a time between j+1 and j+2.

We tried to fit curves of various forms to these numerical solutions. The most interesting of the results that we obtained are shown in Fig. 7. In that figure, we have graphed our

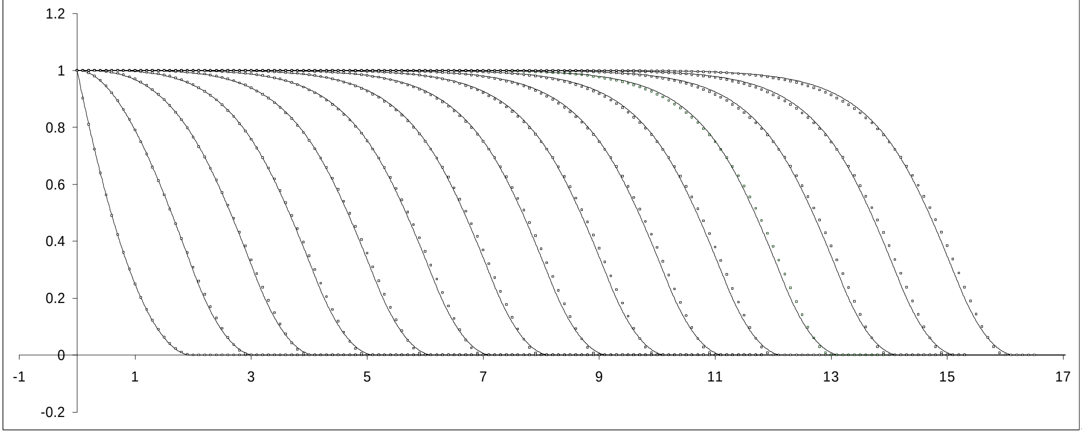

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Figure 7. Numerical solution (solid curves) and conjectured approximate solutions $(1-(\frac{t}{j+1})^j)^2$ (dotted curves)

numerical solutions as solid curves, along with the functions $(1-(\frac{t}{j+1})^j)^2$, whose graphs are shown as dotted curves. We can see in the figure that $y_j(t) \approx (1-(\frac{t}{j+1})^j)^2$. In fact, the functions $(1-(\frac{t}{j+1})^j)^2$ fit the solutions of the ODE system quite well.

However, except for the case $j=1$, $y_j(t)$ is not identically equal to $(1-(\frac{t}{j+1})^j)^2$. Our conjecture, based on Fig. 7, is that we can define a rigorous sense in which $\left|y_j(t)-(1-(\frac{t}{j+1})^j)^2\right|$ is small, that is, a rigorous sense in which the functions $(1-(\frac{t}{j+1})^j)^2$ are approximate solutions of the ODE system (4).

A differential-delay equation for the asymptotic profile

Our numerical solutions support the common-sense conjecture that the functions $y_j(t)$ approach shifted versions of the same function as j becomes large. That is, it appears that there is a function $Y(t)$ such that

$$\frac{dY(t)}{dt} = \sqrt{Y(t-1)} - \sqrt{Y(t)}$$

$$\lim Y(t) = 1$$

Such that $y_j(t) = Y(t+j)$. Here, we have hypothesized not only the form of this differential-delay equation, but that the delay is 1; we base this aspect of the conjecture on our numerical results. More generally we might conjecture that the function $Y(t)$ satisfies an equation of the form $\frac{dY(t)}{dt} = \sqrt{Y(t-t)} - \sqrt{Y(t)}$, for some delay t that may not equal 1.

REFERENCES

1. D. S. Landes, *Revolution In Time*, *Clocks and the Making of the Modern World*, Harvard University Press, Cambridge, 1983.
2. A. Lepschy, G. A. Antonio and U. Viaro, Feedback Control in Ancient Water and Mechanical Clocks, *IEEE Transactions on Education*, **35**, (1992).
3. J. MacLachlan, *Galileo Galilei: First Physicist*, Oxford University Press, New York, 1997.
4. O. Mayr, *The Origins of Feedback Control*, MIT Press, Cambridge, 1970.
5. J. S. McNown, When Time Flowed, The Story of the Clepsydra, La Houille Blanche, 5, 1976, 347-353
6. A. A. Mills, Newton's Water Clocks and the Fluid Mechanics of Clepsydrae,
7. J. V. Noble and D. J. de Solla Price, The Water clock in the Tower of the Winds, Amer. J. Archaeology, **72**, 1968, 345-355
8. A. Pogo, Egyptian Water Clocks, Isis, **25**, (1936), 403-425. Reprinted in *Philosophers and Machines*, O. Mayr, editor, Science History Publications, 1976.
9. J. Stewart, *Calculus*, 4th ed. Brooks/Cole, Pacific Grove, 1999.
10. A. J. Turner, *The Time Museum, Vol. 1, Time Measuring Instruments, Part 3, Water-clocks, Sand-glasses, Fire-clocks*, Rockford, 1984.
11. R. P. Feynman, R. B. Leighton, and M. L. Sands, *The Feynman Lectures on Physics, V. II*, Addison-Wesley, Reading, 1964.
12. M. W. Hirsch, S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press, New York, 1974.
13. http://physics.nist.gov/GenInt/Time/early.html
14. http://www.town.middleton.ns.ca/discover/clock.html
15. http://www.marcdatabase.com/~lemur/dm-gitton.html