Impulsing the Pendulum: Escapement Error

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I. INTRODUCTION

The purpose of this article is to investigate the following questions: What is the ideal location to impulse a pendulum? What are the consequences of not impulsing at the ideal location from the viewpoint of accurate timekeeping?

Over the years a lot of attention has been given to circular error and not enough to the so-called "escapement" error, which is actually impulse error. Impulse error is the change in the period of a pendulum due to the act of impulsing. Impulsing is required to maintain the amplitude of the pendulum and to take a synchronizing signal from it.

In our space-time continuum three dimensions are required to define a point. The ideal location for impulsing a pendulum is defined by the swing plane, the center of percussion, and the position of equilibrium of the pendulum. The position of equilibrium is the position the pendulum takes when it is at rest.

II. CIRCULAR ERROR

An item to consider for precision timekeeping with a pendulum is maintaining a constant amplitude of swing. Most present-day pendulums follow a circular path, so they have what is called circular error.

Circular error is the change in the period of a pendulum due to the fact that a freely swinging pendulum, following a circular path under the influence of gravity, takes longer to traverse a large arc than a small one.

As can be seen in Table 1, the rate of change of circular error increases with an increased amplitude. Thus, to minimize the problems with circular error, it is best to keep the amplitude of the pendulum as small as possible. Later we will use the results of Table 1 to try to cancel out the effect of the change in circular error with the change in the impulse error. It should also be noted that in both cases (i.e., circular and impulse errors), the "error" itself causes no problem in timekeeping because the pendulum length can be adjusted to compensate for them; the real problem is the change in the error due to slight variations in the amplitude of the pendulum.

Table 1. Circular Error for 1-Second Nondamped Pendulum, Series Solution

Length = 99.3961 cm. Nominal Period = 2.0000 sec.

g = 981.0000 cm/sec ^{2} Cycles/day = 43200.0

1	2	3	4	5	6	7	8
Semi-Amplitude		Actual	Circ.	Lin. of	Rate of	Lin. of	Max.
θ_o degrees	θ_o minutes	Period sec.	Error sec/day	Circ. Error	Change sec/day/asec	Rates of Change	Velocity cm/sec
0.017	1.00	2.000000	0.0005	1.0	1.5231E-05	1.00000	0.09083
0.250	15.00	2.000002	0.1028	15.0	2.2846E-04	1.00000	1.362
0.500	30.00	2.000010	0.4112	30.0	4.5693E-04	1.00001	2.725
1.000	60.00	2.000038	1.6450	60.0	9.1388E-04	1.00003	5.450
2.000	120.00	2.000152	6.5802	120.0	1.8280E-03	1.00014	10.90
3.000	180.00	2.000343	14.807	180.0	2.7424E-03	1.00031	16.35
4.000	240.00	2.000609	26.326	240.1	3.6575E-03	1.00056	21.80
5.000	300.00	2.000952	41.141	300.1	4.5733E-03	1.00087	27.24
6.000	360.00	2.001372	59.255	360.2	5.4900E-03	1.00126	32.69
7.000	420.00	2.001867	80.671	420.4	6.4079E-03	1.00171	38.13
8.000	480.00	2.002440	105.39	480.5	7.3272E-03	1.00224	43.56
9.000	540.00	2.003089	133.43	540.8	8.2480E-03	1.00283	49.00
10.000	600.00	2.003814	164.78	601.0	9.1705E-03	1.00350	54.43

DESCRIPTION OF TERMS

Nominal period - The desired period of a pendulum in seconds, the period is twice the time of the pendulum designation (i.e., for a 1-second pendulum the period is 2 seconds).

g - Acceleration due to gravity, in centimeters per second per second.

Length - Theoretical pendulum length for the nominal period, in centimeters. See Eq. (B-1).

1 & 2. Semiamplitude - The chosen angle of swing from the position of equilibrium to the position of maximum displacement, given in degrees and minutes.

3. Period - The actual period of the pendulum, including the circular error. See Eq. (E-3).

4. Circular error - The increase in the period of a circular pendulum due to an increase in the amplitude of swing, given in seconds per day. See Eq. (E-2).

5. Linearity of circular error - Compared to the 1-minute value.

6. Rate of Change - The rate of change of the circular error, which is the slope of the circular error vs. semiamplitude curve, in seconds per day per second of arc. See Eq. (E-4).

7. Linearity of rates of change - Rate of change of circular error compared to the semi amplitude of the pendulum.

8. Maximum velocity - The maximum velocity of the pendulum, which occurs at the position of equilibrium, in centimeters per second. See Eq. (B-3).

III. SWING PLANE

The impulsing force should be perpendicular to the axis of the pendulum and in the swing plane of the pendulum. The axis of the pendulum is the line perpendicular to its axis of rotation and passing through its center of mass. The swing plane is the plane generated by rotating the axis of the pendulum about the axis of rotation.

Any force that is not in the swing plane and perpendicular to the axis of the pendulum produces one or more of the following effects: twisting, bending, and stretching or shortening of the pendulum. These effects cause extraneous movements of the pendulum that affect the regularity of its motion.

IV. CENTER OF PERCUSSION (CENTER OF OSCILLATION)

A ball hitting the "sweet spot" of a baseball bat or tennis racket causes no sting or jarring to the hands gripping the bat or racket. The sweet spot is at the center of percussion of the bat or racket.

If a pendulum is impulsed at its center of percussion, there is no reaction at the pivot point or center of rotation. For a compound pendulum (a real-life pendulum) the distance from the center of rotation to the center of percussion is equal to the length of a theoretical simple pendulum of the same period. Another relationship between the center of rotation and the center of percussion is that they are conjugate to each other. That is, if the pendulum is suspended from its center of percussion, its period remains the same, and the original center of rotation becomes the new center of percussion. For this reason it is also called the "center of oscillation."

Any impulsing force applied at other than the center of percussion will disturb the pendulum by causing the pendulum rod to flex and vibrate. The closer the location of the impulsing is to the center of percussion the lighter the required pendulum rod.

V. "Q" IS FOR QUALITY?[1]

Q is called the Quality factor and is a measure of the efficiency of a pendulum in maintaining its amplitude. From the literature it would seem that a "free" pendulum requiring no impulsing and therefore having an infinite Q would be the perfect pendulum. But from the timekeeping point of view this is not true.

It is true that "other things being equal" the higher the Q, the more constant the period of the pendulum, but the corollary that any pendulum with a high Q has a more constant period than any other pendulum with a lower Q is untrue.

Q is a measure of the relative magnitude of the impulse required to maintain pendulum amplitude; thus, it is also a measure of the magnitude of the various energy losses the pendulum experiences. There are many factors, not measured by Q, that affect the timekeeping ability of a pendulum. For example, Q would not be directly affected by temperature error, errors due to buoyancy effects, circular error, or errors due to phase angle changes in the impulsing.

From the above we see that although it is desirable and maybe even necessary to have a large Q for good timekeeping, it is not sufficient. A pendulum with a high Q can be a good or bad timekeeper depending on its attributes not measured by Q, but a pendulum with a low Q will never be a great timekeeper.

VI. "ESCAPEMENT" - IMPULSE ERROR[2,3,4,5,6,7]

Escapement error was first described by George Biddell Airy as Example 7 in his 1826 article "On the Disturbances of Pendulums and Balances, and on the Theory of Escapements."

There have been qualitative discussions of the problem since then, but no exact and comprehensive quantitative presentations, due to the extensive computations required.

VII. IMPULSE ERROR, QUALITATIVE REVIEW

To understand the cause of impulse error we will examine five cases of impulsing the pendulum:

A. Impulse at the position of equilibrium

B. Impulse at the end of swing, away from the position of equilibrium

C. Impulse at the end of swing, toward the position of equilibrium

D. Impulse at intermediate phase angle, away from the position of equilibrium

E. Impulse at intermediate phase angle, toward the position of equilibrium

For this exercise use a 1-second isochronous pendulum (i.e., no circular error).

A. The velocity of the pendulum is increased by the impulse at the position of equilibrium. This increases the semiamplitude of the pendulum, but there is no change in the period because the pendulum is isochronous.

B. The pendulum takes 0.5 second to travel from the position of equilibrium to the end of the swing, at which point it is impulsed in the direction away from the position of equilibrium. This causes the pendulum to travel some additional distance from the position of equilibrium, before stopping again, which takes some additional time; let's call it Δt_B. After the pendulum reaches this point, it reverses direction and returns to the position of equilibrium in 0.5 second because it is isochronous. Thus, the total time of travel is 1.0 second + Δt_B. The term + Δt_B is the impulse error for this case.

C. The pendulum takes 0.5 second to travel from the position of equilibrium to the end of the swing, at which point it is impulsed in the direction toward the position of equilibrium. This causes the pendulum to start traveling toward the position of equilibrium at some finite velocity rather than zero speed, as if it had started from some point further from the position of equilibrium. Thus, the time to return to the position of equilibrium is 0.5 second - Δt_C, and the total travel time is 1.0 second - Δt_C. The term - Δt_C is the impulse error for this case.

D. The pendulum swings from the position of equilibrium to the impulse phase angle, at which point it is impulsed in the direction away from the position of equilibrium and continues to the end of the swing.

After the pendulum reaches this end point, it reverses direction. Because the pendulum is isochronous, it returns to the position of equilibrium in 0.5 second However, the initial travel from the position of equilibrium to the end of the swing took 0.5 second + Δt_D because the semiamplitude traveled was for the pendulum with the impulsed energy state; however, initially the pendulum was traveling at the lower preimpulse velocity until it reached the point of impulsing. Thus, the total time of travel is 1.0 second. + Δt_D. The term + Δt_D is the impulse error for this case.

E. The pendulum swings from the position of equilibrium to the end of the swing in 0.5 second After reaching this point the pendulum reverses direction and continues to the impulse phase angle, at which point it is impulsed in the direction toward the position of equilibrium and returns to the position of equilibrium. The trip from the end of the swing to the position of equilibrium took less than 0.5 second by - Δt_E because the semiamplitude traveled was for the pendulum with the preimpulsed energy state; however, after the impulse the pendulum was traveling at the higher postimpulse velocity from the point of impulsing until it reached the position of equilibrium. Thus, the total time of travel is 1.0 second - Δt_E; the term - Δt_E is the impulse error for this case.

Even though the above discussion was done for a 1.0 second isochronous pendulum, the conclusions are valid for all pendulums, balance wheels, and torsion pendulums.

Impulse (escapement) error is due to the time difference in the travel of the pendulum from the position of equilibrium to the point of impulse, with the initial and impulsed energy states.

All interference to the motion of a pendulum is a form of impulsing. There is both positive and negative impulsing. Positive impulsing (in the direction of motion of the pendulum) is energy transferred to the pendulum to maintain its energy level, whereas negative impulsing (opposed to the direction of motion of the pendulum) is energy taken from the pendulum.

It should be noted that any forces applied to the pendulum and acting toward the position of equilibrium reduce the period of the pendulum, whereas forces acting away from the position of equilibrium increase the period.

Because damping forces act throughout the motion of a pendulum and are generally symmetric (i.e., acting half the time toward the position of equilibrium and the other half away from the position of equilibrium), they have practically no net effect on the period of the pendulum.

However, the positive impulsing force generally acts repetitively over a small portion of the pendulum's motion and usually in the same direction, either toward or away from the position of equilibrium, depending on the escapement, therefore producing an impulse error.

VIII. IMPULSE ERROR, QUANTITATIVE EVALUATION

The values of the impulse errors and rates of change were calculated on a Quattro Pro spreadsheet by using the equations derived in the Appendices. The Quattro Pro spreadsheet carries 15 significant figures. It would have been better to use a program that carries more significant figures, such as FORTRAN, but with care the Quattro Pro program was adequate for the job.

First, the impulse errors and rates of change were calculated for the following four cases (see Table 2):

1. <u>Nondamped circular path pendulum</u> using a series solution of elliptic integrals. This was the first case used to calculate impulse error; the equation is the only pendulum equation of motion that has an exact solution and is the easiest to use. It is the classic equation of motion of a pendulum following a circular path. The one deficiency of this equation is that it is a mathematical equivalent to a perpetual motion machine. Because there is no damping to reduce the amplitude of swing, any impulse will increase the amplitude of the pendulum. There was a question whether this would affect the results.
2. <u>Linearly damped circular path pendulum</u> using fourth-order Runge-Kutta method. This case includes damping and a reduction in amplitude, which is countered by impulsing. But this case requires numerical integration for solution and thus a much greater amount of calculation.
3. <u>Nondamped cycloidal path pendulum</u> using numerical linear integration
4. <u>Linearly damped cycloidal path pendulum</u> using an algebraic solution. Cases 3 and 4 are for a cycloidal path pendulum, which is isochronous and thus has no circular error and were used to ensure that circular error was correctly eliminated in cases 1 and 2.

These calculations were done for Q = 10,000 and semiamplitude = 240 minutes. All cases were calculated at the same energy levels, and the circular path solutions were corrected for circular error. The calculations for the four cases used totally different mathematical techniques; however, the results came out vir-

Table 2. Comparison of Impulse Error for Circular and Cyclodial Pendulums, Both Linearly Damped and Nondamped

Path	Semi-Amplitude		Q =	10,000.0	Length =	99.3961	cm.
Circle	θo =	240.0000 minutes			a =	24.8490	cm.
Cycloid	βo =	240.1465 minutes			g =	981.00	cm/sec [2]

Path	Circular				Cycloidal				
Damping	Non-Damped		Linearly Damped		Linearly Damped		Non-Damped		
Imp. Loc. θ minutes	Imp. Err ±sec./day	Rt. Chng. Imp. Err. sec/day/asec	Imp. Err. ±sec./day	Rt. Chng. Imp. Err. sec/day/asec	Imp. Err. ±sec./day	Rt. Chng. Imp. Err. sec/day/asec	Imp. Err. ±sec./day	Rt. Chng. Imp. Err. sec/day/asec	Imp. Loc. β minutes
0.0000	0.0000	2.9990E-04	0.0000	3.0036E-04	0.0000	3.0004E-04	0.0000	2.9999E-04	0.0000
1.0000	0.017994	2.9991E-04	0.018022	3.0037E-04	0.018002	3.0005E-04	0.018000	3.0000E-04	1.0000
72.0000	1.3581	3.4548E-04	1.3601	3.4594E-04	1.3587	3.4557E-04	1.3586	3.4551E-04	72.0039
144.0000	3.2390	5.8579E-04	3.2430	5.8626E-04	3.2403	5.8555E-04	3.2398	5.8546E-04	144.0316
216.0000	8.9162	3.6193E-03	8.9229	3.6206E-03	8.9178	3.6138E-03	8.9166	3.6132E-03	216.1068
223.2000	10.925	6.0335E-03	10.932	6.0355E-03	10.926	6.0235E-03	10.931	6.0226E-03	223.3178
230.4000	14.798	1.3632E-02	14.807	1.3636E-02	14.800	1.3607E-02	14.803	1.3605E-02	230.5296
237.6000	30.206	0.10567	30.219	0.10571	30.205	0.10547	30.206	0.10545	237.7422
238.3200	36.125	0.17913	36.141	0.17920	36.124	0.17879	36.124	0.17876	238.4635
239.0400	47.702	0.40871	47.721	0.40891	47.699	0.40798	47.698	0.40792	239.1848
239.7600	93.032	3.0079	93.066	3.0169	93.023	3.0099	93.018	3.0094	239.9061
240.0000	487.55	226.24	487.56	226.124	487.41	225.98	487.60	226.19	240.1465

Imp. Loc. = Impulse Location Imp. Err. = Impulse Error sec/day/asec = seconds per day per arc second
Rt. Chng. Imp. Err. = Rate of Change of Impulse Error

tually identical (see Table 2). Because of this a decision was made to perform all the rest of the calculations by using the equations for the nondamped circular path pendulum because it was the simplest to work with.

The impulse errors and rates of change of the impulse error (RCIE) were calculated for five values of the quality factor Q, from Q = 100 to Q = 1,000,000. For each value of Q, calculations were done for five values of the semiamplitude, from 30 minutes to 480 minutes of arc, which covers the range for most presently used pendulums.

The impulse is provided once per swing by increasing the velocity at the required point in the swing.

From these results it was found that impulse error (see Figure 1) is a function of two variables: (1) the quality factor Q, which defines the magnitude of the impulse, and (2) the relative impulse location. The relative impulse location is obtained by dividing the phase angle of the impulse by the semiamplitude of the pendulum (phase angle/semiamplitude).

Because of this, even though the impulse error is independent of the semiamplitude, the RCIE is inversely proportional to the semiamplitude (see Figure 2 and Table 3). We see this by noting that for any of the lines of data in Table 3, for a particular relative impulse location (column 1) the impulse error (column 2) is constant for all semiamplitudes, whereas the RCIE (columns 4, 6, 8, 10, and 12) is reduced with an increase in semiamplitude. That is, the greater the semiamplitude the smaller the RCIE. We also see that the RCIE grows very rapidly near the very end of the swing. This is the reason that pendulums impulsed there are very poor timekeepers.

During 90 percent of the swing of the pendulum, the impulse error is nearly linearly inversely proportional to Q with a deviation of less than 5 percent; however, near the end of the swing the relation becomes nonlinear and changes to a relation of the square root of the inverse of Q at the very end of the swing.

The reason for the above is as follows: The impulse, no matter where in the swing it is regularly applied, has to provide the same amount of energy to maintain the amplitude of the pendulum. The moving pendulum has two forms of energy: potential and kinetic. The potential energy is due to the location of the center of mass above some fixed datum, whereas the kinetic energy is due to the motion of the pendulum and is a function of the square of the velocity of the center of mass of the pendulum. The impulse increases the kinetic energy of the pendulum. When the pendulum is traveling at a relatively high velocity (e.g., near the position of equilibrium), the change in velocity to provide the impulse energy is very small, and it has a very small effect on the period of the pendulum. But when the pendulum is at or near the ends of the swing (i.e., moving very slowly), the change in velocity is much greater and so is its effect on the period of the pendulum. The kinetic energy of the pendulum, as stated earlier, is proportional to the square of the velocity of the pendulum, whereas the period of the pendulum is proportional to the velocity. This is the cause for the impulse error at the ends of the swing being proportional to the inverse of the square root of Q.

For over 90 percent of the swing, mentioned earlier, the average impulse error is essentially the same whether the escapement impulses the pendulum once

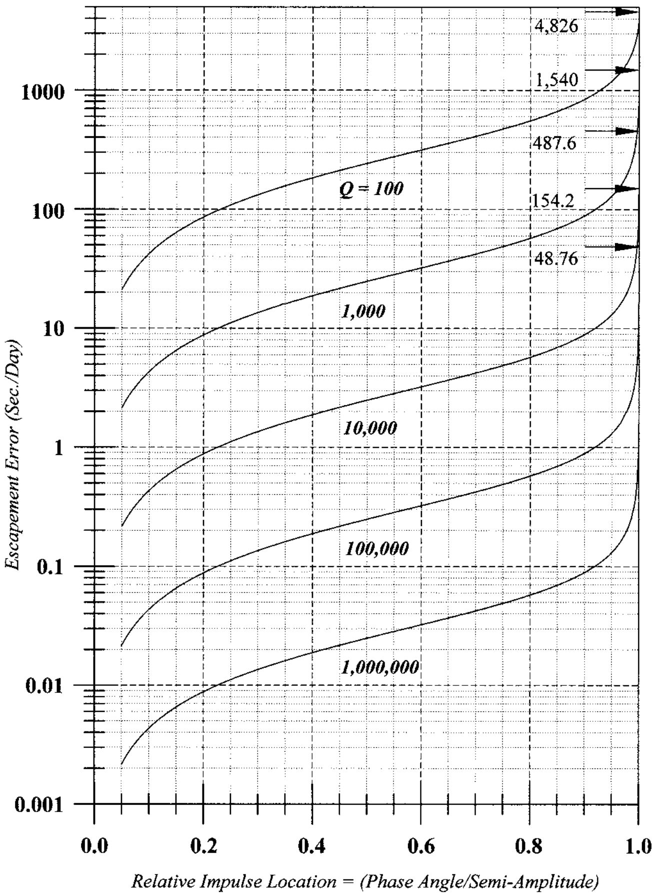

Figure 1. Escapement error for one impulse/swing.

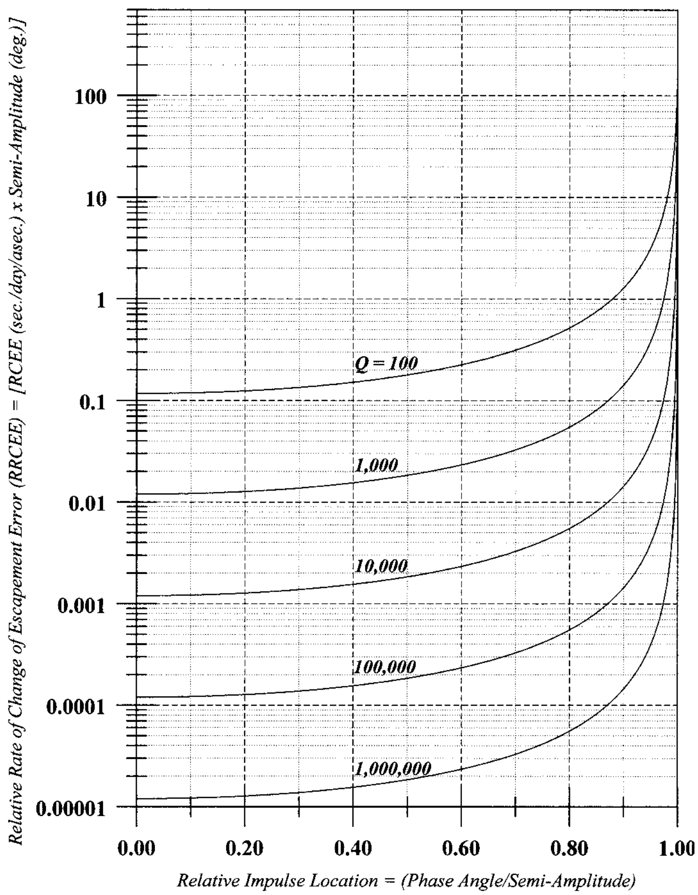

Figure 2. Relative rate of change of escapement error.

in every swing, as in these calculations, or once every other swing, or even once in 30 swings as in some escapements. It is important to note that even though the average impulse error is practically the same, the actual impulse error only occurs in the swing that the impulse is applied.

Impulse error can be positive or negative, depending on the direction of the impulsing force. An impulsing force acting away from the position of equilibrium produces a positive impulse error, whereas one toward the position of equilibrium produces a negative impulse error. A positive impulse error increases the period of the pendulum, but a negative one reduces the period.

Table 3A. Impulse Error for 1-Second Nondamped Circular Pendulum, Eliptic Integral - Series Solution

Length = 99.3961 cm. g = 981.00 cm./sec.^{2}

Semi-Amplitude θ_o =		30.0		60.000		120.000		240.000		480.000	
Relative Impulse Loc.	Imp. Err. ±sec/day	Imp. Loc. θ minutes	Rt. Chng. Imp. Err. sc/day/asc	Imp. Loc. θ minutes	Rt. Chng. Imp. Err. sc/day/asc	Imp. Loc. θ minutes	Rt. Chng. Imp. Err. sc/day/asc	Imp. Loc. θ minutes	Rt. Chng. Imp. Err. sc/day/asc	Imp. Loc. θ minutes	Rt. Chng: Imp. Err. sc/day/asc
Q =	100										
0.0000	0.000	0.000	0.2345	0.000	0.1172	0.000	0.05862	0.000	0.02931	0.000	0.01465
0.3000	132.6	9.000	0.2695	18.000	0.1348	36.000	0.06738	72.000	0.03369	144.000	0.01684
0.6000	315.2	18.000	0.4522	36.000	0.2261	72.000	0.1130	144.000	0.05653	288.000	0.02827
0.9000	844.9	27.000	2.582	54.000	1.291	108.000	0.6455	216.000	0.3228	432.000	0.1616
0.9300	1021	27.900	4.126	55.800	2.063	111.600	1.032	223.200	0.5160	446.400	0.2583
0.9600	1337	28.800	8.461	57.600	4.231	115.200	2.116	230.400	1.058	460.800	0.5296
0.9900	2295	29.700	40.86	59.400	20.43	118.800	10.22	237.600	5.109	475.200	2.557
0.9930	2556	29.790	57.62	59.580	28.81	119.160	14.41	238.320	7.205	476.640	3.605
0.9960	2950	29.880	94.02	59.760	47.01	119.520	23.51	239.040	11.76	478.080	5.882
0.9990	3748	29.970	258.1	59.940	129.1	119.880	64.54	239.760	32.27	479.520	16.14
1.0000	4826	30.000	831.3	60.000	605.3	120.000	436.9	240.000	313.4	480.000	223.9

1 - Relative Impulse Loc. = Relative Impulse Location from position of equilibrium = 0.0000 to Semiamplitude = 1.0000

2 - Imp. Err. = Impulse Error in ±Seconds per Day, + is impulse toward position of equilibrium, - is away from position of equilibrium

3, 5, 7, 9, & 11 - Imp. Loc. = Impulse Location in minutes.

4, 6, 8, 10, & 12 - Rt. Chng. Imp. Err. = Rate of Change of Impulse Error in Seconds per Day per Arc Second.

Table 3B. Impulse Error for 1-Second Nondamped Circular Pendulum, Eliptic Integral - Series Solution

Semi-Ampl. θ_o =		30.0		60.000		120.000		240.000		480.000	
Relative Impulse Loc.	Imp. Err. ±Sec./Day	Imp. Loc. θ minutes	Rt. Chng. Imp. Err. sc/day/asc	Imp. Loc. θ minutes	Rt. Chng. Imp. Err. sc/day/asc	Imp. Loc. θ minutes	Rt. Chng. Imp. Err. sc/day/asc	Imp. Loc. θ minutes	Rt. Chng. Imp. Err. sc/day/asc	Imp. Loc. θ minutes	Rt. Chng. Imp. Err. sc/day/asc
Q =	1,000										
0.0000	0.000	0.000	0.02394	0.000	0.01972	0.000	0.005986	0.000	0.002993	0.000	0.001496
0.3000	13.55	9.000	0.02758	18.000	0.01379	36.000	0.006894	72.000	0.003447	144.000	0.001723
0.6000	32.31	18.000	0.04670	36.000	0.02335	72.000	0.01168	144.000	0.005839	288.000	0.002920
0.9000	88.71	27.000	0.2862	54.000	0.1431	108.000	0.07157	216.000	0.03579	432.000	0.01792
0.9300	108.5	27.900	0.4750	55.800	0.2375	111.600	0.1188	223.200	0.05941	446.400	0.02974
0.9600	146.5	28.800	1.062	57.600	0.5308	115.200	0.2654	230.400	0.1327	460.800	0.06646
0.9900	291.7	29.700	7.654	59.400	3.827	118.800	1.914	237.600	0.9572	475.200	0.4793
0.9930	344.4	29.790	12.49	59.580	6.246	119.160	3.123	238.320	1.562	476.640	0.7821
0.9960	441.2	29.880	26.14	59.760	13.07	119.520	6.536	239.040	3.269	478.080	1.636
0.9990	740.7	29.970	128.6	59.940	64.32	119.880	32.16	239.760	16.08	479.520	8.049
1.0000	1540	30.000	664.8	60.000	517.5	120.000	391.7	240.000	290.4	480.000	212.3
Q =	10,000										
0.0000	0.000	0.000	0.002399	0.000	0.001200	0.000	0.0005998	0.000	0.0002999	0.000	0.0001499
0.3000	1.358	9.000	0.002764	18.000	0.001382	36.000	0.0006910	72.000	0.0003455	144.000	0.0001727
0.6000	3.239	18.000	0.004686	36.000	0.002343	72.000	0.001171	144.000	0.0005858	288.000	0.0002930
0.9000	8.916	27.000	0.02894	54.000	0.01447	108.000	0.007236	216.000	0.003619	432.000	0.001812
0.9300	10.92	27.900	0.04825	55.800	0.02412	111.600	0.01206	223.200	0.006033	446.400	0.003021
0.9600	14.80	28.800	0.1090	57.600	0.05450	115.200	0.02725	230.400	0.01363	460.800	0.006825
0.9900	30.21	29.700	0.8449	59.400	0.4225	118.800	0.2113	237.600	0.1057	475.200	0.05291
0.9930	36.16	29.790	1.432	59.580	0.7162	119.160	0.3581	238.320	0.1791	476.640	0.08970
0.9960	47.70	29.880	3.268	59.760	1.634	119.520	0.8171	239.040	0.4087	478.080	0.2047
0.9990	93.04	29.970	24.05	59.940	12.03	119.880	6.014	239.760	3.008	479.520	1.506
1.0000	487.6	30.000	365.9	60.000	324.6	120.000	276.7	240.000	226.2	480.000	178.1

IX. IMPULSE ERROR, CONCLUSIONS

By comparing the rate of change of the circular error (RCCE) (see Table 1) with the RCIE (see Table 3) we see that the optimum semiamplitude is dependent on the value of Q for the clock and the relative impulse location because increasing the amplitude increases the RCCE but decreases the RCIE.

The ideal would be to eliminate or at least to minimize the total change in the circular error plus impulse error by matching the RCCE and RCIE by adjusting the semiamplitude for a pendulum with a specific value of Q.

We can theoretically eliminate the combined change in circular error plus the impulse error. To do this we would need an escapement with two attributes: (1) The impulse must be toward the position of equilibrium and (2) The relative impulse location must be adjustable.

Then we could choose a combination of amplitude and phase angle (relative impulse location) such that when the amplitude varied, the increase or decrease in the circular error would be corrected by the decrease or increase in the impulse error.

For example from Table 1 the RCCE for a semiamplitude of 4° is 0.00366 sec/day/asec, whereas from Table 3B for a Q = 10,000, a semiamplitude of 240' (4°) for an impulse toward the position of equilibrium and a phase angle of 216' the RCIE is -0.00362 sec/day/asec;

Table 3C. Impulse Error for 1-Second Nondamped Circular Pendulum, Eliptical Integral - Series Solution

Semi-Ampl. θ_o =		30.0		60.000		120.000		240.000		480.000	
Relative Impulse Loc.	Imp. Err. ±Sec./Day	Imp. Loc. θ minutes	Rt. Chng. Imp. Err. sc/day/asc	Imp. Loc. θ minutes	Rt. Chng. Imp. Err. sc/day/asc	Imp. Loc. θ minutes	Rt. Chng. Imp. Err. sc/day/asc	Imp. Loc. θ minutes	Rt. Chng. Imp. Err. sc/day/asc	Imp. Loc. θ minutes	Rt. Chng. Imp. Err. sc/day/asc
Q =	100,000										
0.0000	0.000	0.000	0.0002400	0.000	0.0001200	0.000	6.000E-05	0.000	3.000E-05	0.000	1.499E-05
0.3000	0.1358	9.000	0.0002765	18.000	0.0001382	36.000	6.912E-05	72.000	3.456E-05	144.000	1.728E-05
0.6000	0.3240	18.000	0.0004687	36.000	0.0002344	72.000	0.0001172	144.000	5.860E-05	288.000	2.931E-05
0.9000	0.8921	27.000	0.002898	54.000	0.001449	108.000	0.0007245	216.000	0.0003623	432.000	0.0001814
0.9300	1.093	27.900	0.004832	55.800	0.002416	111.600	0.001208	223.200	0.0006043	446.400	0.0003025
0.9600	1.481	28.800	0.01093	57.600	0.005465	115.200	0.002733	230.400	0.001367	460.800	0.0006844
0.9900	3.032	29.700	0.08539	59.400	0.04270	118.800	0.02135	237.600	0.010679	475.200	0.005348
0.9930	3.631	29.790	0.1454	59.580	0.07270	119.160	0.03636	238.320	0.018184	476.640	0.009106
0.9960	4.813	29.880	0.3354	59.760	0.1677	119.520	0.08387	239.040	0.041950	478.080	0.02101
0.9990	9.620	29.970	2.654	59.940	1.327	119.880	0.6636	239.760	0.3319	479.520	0.1662
1.0000	154.2	30.000	141.3	60.000	136.1	120.000	128.9	240.000	119.4	480.000	107.2
Q =	1.0E+06										
0.0000	0.000	0.000	2.400E-05	0.000	1.200E-05	0.000	6.000E-06	0.000	3.000E-06	0.000	1.499E-06
0.3000	0.01358	9.000	2.765E-05	18.000	1.382E-05	36.000	6.912E-06	72.000	3.456E-06	144.000	1.728E-06
0.6000	0.03240	18.000	4.687E-05	36.000	2.344E-05	72.000	1.172E-05	144.000	5.860E-06	288.000	2.931E-06
0.9000	0.08921	27.000	0.0002898	54.000	0.0001449	108.000	7.245E-05	216.000	3.624E-05	432.000	1.814E-05
0.9300	0.1093	27.900	0.0004833	55.800	0.0002417	111.600	0.0001208	223.200	6.044E-05	446.400	3.026E-05
0.9600	0.1482	28.800	0.001093	57.600	0.0005466	115.200	0.0002733	230.400	0.0001367	460.800	6.845E-05
0.9900	0.3033	29.700	0.008548	59.400	0.004274	118.800	0.002137	237.600	0.001069	475.200	0.0005353
0.9930	0.3633	29.790	0.01456	59.580	0.007281	119.160	0.003641	238.320	0.001821	476.640	0.0009120
0.9960	0.4817	29.880	0.03363	59.760	0.01682	119.520	0.008409	239.040	0.004206	478.080	0.002106
0.9990	0.9654	29.970	0.2682	59.940	0.1341	119.880	0.06706	239.760	0.03354	479.520	0.01680
1.0000	48.76	30.000	47.45	60.000	46.92	120.000	46.16	240.000	45.11	480.000	43.67

a slight increase in the phase angle or decrease in the semiamplitude will make the two changes exactly cancel out each other.

X. APPENDIX

For the equations used to obtain the numerical results in this article and their derivation with a complete set of results, see the online appendix at www.nawcc.org/pub/bulsupp.htm or contact the NAWCC Library and Research Center.

About the Author

A retired civil engineer by profession, Dr. George Feinstein has spent his career in various phases of mass transportation. He is very active in Electrical Horology Chapter 78 where he has been the co-editor of the chapter newsletter since 1987. He has contributed several articles to that publication, including descriptions of the Fedchenko clock as well as a list of U.S. patents in electrical horology.

APPENDIX

A. CENTER OF PERCUSSION [8]

To locate the center of percussion of a pendulum we use the equations of impulse and momentum, and the fact that the pendulum is constrained to rotate about its center of rotation.

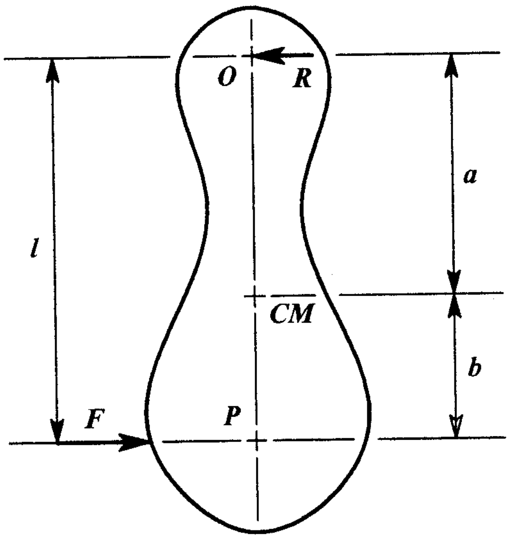

Figure 3.

CM = Center of Mass
O = Center of rotation
P = Center of percussion
I_{cm} = Moment of inertia about $CM = mk^2$
I_o = Moment of inertia about point O
m = Mass of pendulum
k = Radius of gyration of pendulum
F = Impulsing force
R = Impulsing reaction
ℓ = Theoretical pendulum length
Δt = Duration of impulse
v_{cmi} = Linear velocity of the CM at time i
ω_i = Angular velocity at time I
M = Impulsing moment
a = Distance from center of rotation to the center of mass

The equations of impulse and momentum are:

$$\int_{t_1}^{t_2} F\,dt = mv_{cm2} - mv_{cm1} \quad (A-1) \quad and \quad \int_{t_1}^{t_2} M\,dt = I_o\omega_2 - I_o\omega_1 \quad (A-2)$$

applying the above Eq. (A-2) to the impulsing of a pendulum we have:

$$\int_{o}^{\Delta t} M\,dt = \overline{M}\,\Delta t = \overline{F}\,\ell\,\Delta t = I_o\,\omega - I_o\,\omega_o$$

A bar over a term means that it is the average value over the requisite time period. Solving for ω, we obtain:

$$\omega = (\overline{F}\,\ell\,\Delta t)/I_o + \omega_o$$

Now noting that the linear velocity of the *CM* of the pendulum is equal to its angular velocity times its distance from the center of rotation we have the relation:

$$v_{cmi} = \omega_i a \qquad \text{(A-3)}$$

Substituting we have:

$$v_{cm} = [(\overline{F}\,\ell\,\Delta t)/I_o + \omega_o]a$$

Next applying Eq. (A-1) to the pendulum:

$$\int_o^{\Delta t} F\,dt = (\overline{F} - \overline{R})\Delta t = m\,v_{cm} - m\,v_{cmo}$$

Again using Eq. (A-3) we obtain:

$$(\overline{F} - \overline{R})\Delta t = m[(\overline{F}\,\ell\,\Delta t)/I_o + \omega_o]a - m\,\omega_o a$$

Rearranging and simplifying, we have:

$$(\overline{F} - \overline{R})\Delta t = (m\,\overline{F}\,\ell\,\Delta t\,a)/I_o$$

Noting the relations for the moment of inertia:

$$I_{cm} = m\,k^2 \quad and \quad I_o = I_{cm} + m\,a^2 = m(k^2 + a^2)$$

And substituting we get:

$$\overline{R} = \overline{F}[1 - (\ell a)/(k^2 + a^2)]$$

Thus, there is no resultant reaction at the center of rotation if:

$$\ell = a + (k^2/a) \quad and \quad b = (k^2/a) \qquad (A\text{-}4)$$

This is the center of percussion.

B. TRAVEL TIME (CIRCULAR PENDULUM) [9]

Now we will derive the equations determining the time it takes for a pendulum to travel to any point along its circular arc due to the action of gravity.

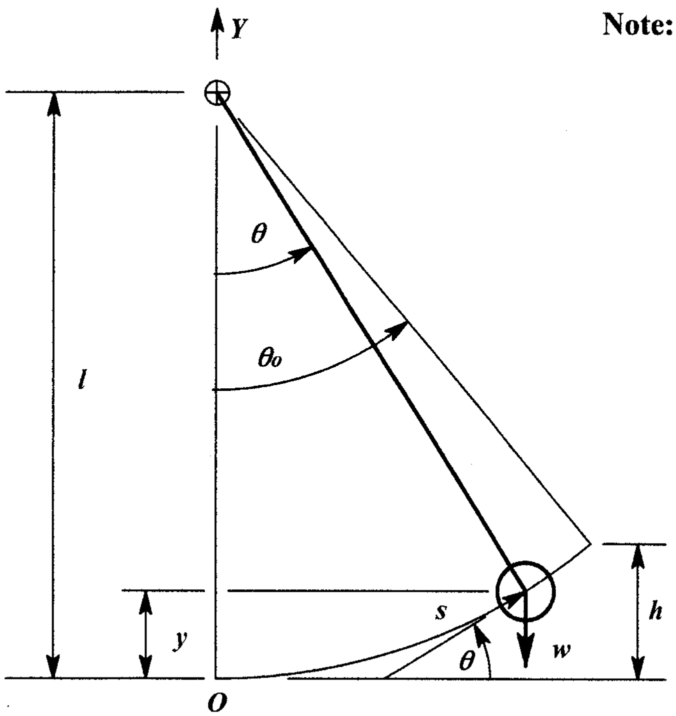

Figure 4.

Note: $s = \ell\theta$ = Distance traveled by the pendulum bob along the circular arc.

$v = ds/dt = \ell d\theta/dt$ = Velocity of the pendulum bob.

$a = dv/dt = d^2s/dt^2 = \ell d^2\theta/dt^2$ = Acceleration of the pendulum bob.

$dv/dt = (dv/ds)(ds/dt) = v\, dv/ds$

$y = \ell(1 - \cos\theta)$, $\sin\theta = dy/ds$

θ_0 = Semiamplitude of pendulum.

$h = \ell(1 - \cos\theta_o)$

$\ell = (P_c^2 \cdot g)/(4\pi^2)$ = Theoretical pendulum length. *(B-1)*

P_c = Nominal period of Circular Pendulum

g = Acceleration due to gravity

w = Weight of pendulum

Applying Newton's Second Law of Motion: $F = ma$ to the pendulum, we get:

$$-w\sin\theta = \frac{w}{g}\ell\frac{d^2\theta}{dt^2}$$

Rearranging and using the above listed relations, we have:

$$v\frac{dv}{ds} = -g\frac{dy}{ds}$$

Multiplying by *ds* we get a simple first order equation, then integrating:

$$\int_o^v v\,dv = -g\int_h^y dy$$

Evaluating the integrals we obtain:

$$\frac{v^2}{2} = g(h - y)$$

energy of the system remains constant, to the pendulum:

$$E = T + V \qquad (B\text{-}2)$$

where: E = Total Energy Stored in the Pendulum, T = Kinetic Energy of the Pendulum, and V = Potential Energy of the Pendulum; and

$$T = \frac{w}{2g}v^2 , \quad and \quad V = w\ell(1 - \cos\theta) = wy$$

Note that at $y = h$, $v = 0$, therefore:

$$\frac{w}{2g}v^2 + wy = wh, \quad and \quad \frac{v^2}{2} = g(h - y)$$

From which we get:

$$v = \frac{ds}{dt} = \pm\sqrt{2g(h-y)} \qquad (B\text{-}3)$$

And noting that:

$$\sin\theta = \frac{dy}{ds} = \frac{\sqrt{\ell^2 - (\ell - y)^2}}{\ell}$$

From which we obtain:

$$ds = \frac{\ell dy}{\sqrt{2\ell y - y^2}}$$

and

$$dt = \pm\frac{\ell\, dy}{\sqrt{2g}\sqrt{h-y}\sqrt{2\ell y - y^2}}$$

Thus the time for the pendulum to go from the point of equilibrium (the lowest point in the arc) to any point along the arc is:

$$\int_o^{T_c} dt = T_c = \frac{\ell}{\sqrt{2g}}\int_o^y \frac{dy}{\sqrt{h-y}\sqrt{2\ell y - y^2}}$$

Now let $y = h\, sin^2\varphi$, then

$$dy = 2h \sin\varphi \cos\varphi \, d\varphi, \quad \textit{and} \quad \sqrt{h-y} = \sqrt{h}\cos\varphi$$

and

$$T_c = \frac{\ell}{\sqrt{2g}} \int_o^{\varphi} \frac{2h \sin\varphi \cos\varphi \, d\varphi}{\sqrt{h}\cos\varphi \sqrt{2\ell h \sin^2\varphi - h^2 \sin^4\varphi}}$$

Simplifying, we get:

$$T_c = \sqrt{\frac{\ell}{g}} \int_o^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} \,, \quad \textit{where} \quad k = \sqrt{h/2\ell} < 1 \qquad (B\text{-}4)$$

The integral is an incomplete elliptic integral of the first kind, and is usually noted as:

$$F(k,\varphi) \equiv \int_o^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} \,, \quad 0 < k < 1 \qquad (B\text{-}5)$$

where:

$$k = \sqrt{h/2\ell} = \sqrt{\frac{1 - \cos\theta_o}{2}} = \sin\frac{\theta_o}{2} \qquad (B\text{-}6)$$

and

$$\sin\varphi = \sqrt{\frac{y}{h}} = \sqrt{\frac{1 - \cos\theta}{1 - \cos\theta_o}} = \frac{\sin(\theta/2)}{\sin(\theta_o/2)} \qquad (B\text{-}7)$$

C. ELLIPTIC EVALUATION [10, 11]

You can evaluate an elliptic integral by looking it up in a book of tables or by the following method, using Eq. (B-5):

$$F(k, \varphi) \equiv \int_o^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\phi}} \qquad (0 < k < 1)$$

This is an elliptic integral of the first kind, it has no closed form solution. However, the term $(1 - k^2 \sin^2\phi)^{-½}$ can be expanded into a series by use of the binomial formula,

$$(a + b)^n = a^n + n a^{n-1} b + \frac{n(n-1)}{2!} a^{n-2} b^2 + \dots$$

This series is valid for any n if $|b/a| < 1$. Now $k < 1$ and $\sin\phi \leq 1$, therefore we have $|b/a| = |k^2 \sin^2\phi| < 1$. The expansion becomes:

$$(1 - k^2 \sin^2\varphi)^{-½} = 1 + \frac{k^2}{2}\sin^2\varphi + \frac{(1\cdot 3)}{(2\cdot 4)} k^4 \sin^4\varphi + \frac{(1\cdot 3\cdot 5)}{(2\cdot 4\cdot 6)} k^6 \sin^6\varphi + \dots \quad (C-1)$$

The individual terms can be integrated and evaluated by means of the following relations:

$$\int d\varphi = \varphi\,, \qquad \int \sin^2\varphi\, d\varphi = \frac{\varphi}{2} - \frac{1}{2}\sin\varphi\cos\varphi = \frac{\varphi}{2} - \frac{\sin 2\varphi}{4}$$

and

$$\int \sin^n\varphi\, d\varphi = -\frac{\sin 2\varphi\, \sin^{n-2}\varphi}{2n} + \frac{n-1}{n}\int \sin^{n-2}\varphi\, d\varphi$$

Performing the above operations, we obtain the following infinite series:

$$F(k,\varphi) = A\varphi - \sin 2\varphi \left[\frac{B_o}{2} + \frac{B_1}{3}\sin^2\varphi + \dots + B_n\left[\frac{(2n)!!}{2\cdot(2n+1)!!}\right]\sin^{2n}\varphi + \dots\right] \quad n = 0,1,2,3\dots \ (C-2)$$

$$where{:}\ A = \frac{2K(k)}{\pi},\quad see\ Eq.(E-1);\quad and\quad B_o = A - 1$$

$$and \qquad B_n = B_{n-1} - \left[\frac{(2n-1)!!}{(2n)!!}\right]^2 k^{2n} \qquad n = 1, 2, 3, \dots$$

Note: $n! = 1 \cdot 2 \cdot 3 \dots \cdot n$, $0! = 1$; $(2n+1)!! = 1 \cdot 3 \cdot 5 \dots \cdot (2n+1)$; $(2n)!! = 2 \cdot 4 \cdot 6 \dots \cdot (2n)$
The series converges rapidly for the normal range of values of pendulum amplitude.

D. IMPULSE ERROR

Combining Eqs. (B-4) and (C-2), the impulse error becomes:

$$\Delta T_i = \sqrt{\frac{\ell}{g}}\,[F(k,\varphi) - F(k_i,\varphi_i)] \qquad (D\text{-}1)$$

and to obtain the Rate of Change of the Impulse Error (RCIE) differentiate with respect to θ, i.e.

$$RCIE = \sqrt{\frac{\ell}{g}}\left[\frac{dF(k,\varphi)}{d\theta} - \frac{dF(k_i,\varphi_i)}{d\theta}\right] \qquad (D\text{-}2)$$

Using Eqs. (C-2)

$$\frac{dF(k,\varphi)}{d\theta} = A\frac{d\varphi}{d\theta} - \sum_{n=o}^{\infty} B_n\left[\frac{(2n)!!}{2\cdot(2n+1)!!}\right]\frac{d(\sin 2\varphi \sin^{2n}\varphi)}{d\varphi}\frac{d\varphi}{d\theta}$$

Where using Eq. (B-7) and noting that $\sin 2\varphi = 2\sin\varphi\cos\varphi = 2\sin\varphi\,(1 - \sin^2\varphi)^{1/2}$ and $\cos 2\varphi = 1 - 2\sin^2\varphi$, from which we get:

$$\frac{d\varphi}{d\theta} = \frac{d}{d\theta}\left[\sin^{-1}\left(\frac{\sin(\theta/2)}{k}\right)\right] = \frac{\cos(\theta/2)}{2\sqrt{k^2 - \sin^2(\theta/2)}}$$

and

$$\frac{d(\sin 2\varphi \sin^{2n}\varphi)}{d\varphi} = 2\,[(n+1)\cos 2\varphi + n]\sin^{2n}\varphi$$
$$= \left[(4n+2) - \frac{(4n+4)\sin^2(\theta/2)}{k^2}\right]\frac{\sin^{2n}(\theta/2)}{k^{2n}} \qquad n = 0,1,2,3,\ldots$$

E. CIRCULAR ERROR [10, 11, 12]

To determine the period of the pendulum evaluate Eq.(B-4) for $\theta = \theta_o$, then $\sin\phi = 1$ and $\varphi = \pi/2$, thus:

$$T_c = 4\sqrt{\ell/g}\int_o^{\pi/2}\frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}} = 4\sqrt{\ell/g}\,K(k) \qquad (0 < k < 1)$$

K(k) is a complete elliptic integral of the first kind. Using the expansion given in Eq.(C-1), we have:

$$K(k) = F(k,\pi/2) = \int_o^{\pi/2}\left(1 + \frac{k^2}{2}\sin^2\varphi + \frac{3k^4}{8}\sin^4\varphi + \frac{5k^6}{16}\sin^6\varphi + \ldots\right)d\varphi$$

Integrating term by term we get:

$$K(k) = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \ldots + \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 k^{2n} + \ldots\right] \qquad (E-1)$$

$$if\ k^2 < 1, \quad n = 1,2,3,\ldots$$

Thus the circular error per period is:

$$\Delta T_c = 2\pi\sqrt{\frac{\ell}{g}}\left[\left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \ldots + \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 k^{2n} + \ldots\right] \qquad (E-2)$$

Noting from Eq. (B-6) that for a pendulum $k = sin(\theta_o/2)$, the period of the pendulum becomes:

$$T_c = 2\pi\sqrt{\frac{\ell}{g}}\left[1 + \left(\frac{1}{2}\right)^2 \sin^2\frac{\theta_o}{2} + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \sin^4\frac{\theta_o}{2} + \ldots + \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \sin^{2n}\frac{\theta_o}{2} + \ldots\right] \quad (E-3)$$

Units: Seconds per Cycle

Further, using the expansion:

$$\sin\frac{\theta_o}{2} = (\theta_o/2) - \frac{(\theta_o/2)^3}{3!} + \frac{(\theta_o/2)^5}{5!} - \frac{(\theta_o/2)^7}{7!} + \ldots$$

We can rewrite the equation for the period in terms of θ_o:

$$T_c = 2\pi\sqrt{\frac{\ell}{g}}\left[1 + \frac{\theta_o^2}{16} + \frac{11\,\theta_o^4}{3{,}072} + \frac{519\,\theta_o^6}{2{,}211{,}840} + \ldots\right]$$

The thing to remember is that the absolute value of the circular error, which is made up of all the terms which include θ_o, can and is corrected for by adjusting the length of the pendulum. The thing that affects the accuracy of the pendulum is the rate of change of the circular error (the slope of the Amplitude vs. Period curve), which measures the error due to small changes in the amplitude of the pendulum. To obtain the rate of change differentiate the equation for the period with respect to the amplitude θ_o, from which we obtain:

$$\frac{dT_c}{d\theta_o} = 2\pi\sqrt{\frac{\ell}{g}}\left[\left(\frac{1}{2}\right)^2 \sin\frac{\theta_o}{2}\cos\frac{\theta_o}{2} + 2\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \sin^3\frac{\theta_o}{2}\cos\frac{\theta_o}{2} + 3\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \sin^5\frac{\theta_o}{2}\cos\frac{\theta_o}{2} + \ldots\right]$$

Noting that $sin(\theta_o/2)cos(\theta_o/2) = (½)sin\ \theta_o$, therefore:

$$\frac{dT_c}{d\theta_o} = \pi\sqrt{\ell/g}\sin\theta_o\left[\left(\frac{1}{2}\right)^2 + 2\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\sin^2\frac{\theta_o}{2} + \ldots + n\left(\frac{(2n-1)!!}{(2n)!!}\right)^2\sin^{2(n-1)}\frac{\theta_o}{2} + \ldots\right], \quad n = 1,2,3,\ldots \qquad (E\text{-}4)$$

Units: Seconds per Cycle per Radian

F. IMPULSIVE "Q" [7, 13]

According to the 3rd Edition of A. L. Rawlings "The Science of Clocks & Watches" $Q = \pi(E/\Delta E)$, where E = total stored energy of the oscillating pendulum, while ΔE = additional energy per swing.

$$\Delta E = IMPULSE = \pi E/Q$$

In the following the subscript i refers to terms associated with the impulse.
For a circular pendulum using the above equation and Figure 4 in Appendix B, we have:

$$E = wh = w\ell(1 - \cos\theta_o)$$

and

$$\Delta E = w(h_i - h) = w\ell(\cos\theta_o - \cos\theta_i) = \frac{\pi}{Q}w\ell(1 - \cos\theta_o)$$

Therefore:

$$\cos\theta_i = (1 + \pi/Q)\cos\theta_o - \pi/Q \qquad (F\text{-}1)$$

For a cycloidal pendulum, see Figure 5 in Appendix H:

$$E = wh = 2aw\sin^2\beta_0$$

and

$$\Delta E = 2aw(\sin^2\beta_i - \sin^2\beta_o) = \frac{\pi}{Q}2aw\sin^2\beta_o$$

Therefore:

$$\sin^2\beta_i = (1 + \pi/Q)\sin^2\beta_o$$

G. DAMPED CIRCULAR PENDULUM [14, 15, 16, 17]

Assuming the resistive (damping) force is proportional to the velocity along the arc. The resistive force is therefore: $2mk\dot{s} = 2mk\ell\dot{\theta}$ and the equation of motion becomes:

$$ml\ddot{\theta} = -2mk\ell\dot{\theta} - mg\sin\theta$$

or

$$\ddot{\theta} + 2k\dot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

Note:

$$s = \ell\theta, \quad \frac{ds}{dt} = \dot{s} = \ell\frac{d\theta}{dt} = \ell\dot{\theta}, \ and \quad \frac{d^2s}{dt^2} = \ddot{s} = \ell\frac{d^2\theta}{dt^2} = \ell\ddot{\theta}$$

$$2k = \frac{1}{\tau} = \frac{\omega_o}{Q}, \quad and \quad \omega_o^2 = \frac{g}{\ell}$$

Thus:

$$\ddot{\theta} + \frac{1}{\tau}\dot{\theta} + \omega_o^2\sin\theta = 0$$

This can be written as two first order differential equations:

$$\dot{\theta} = \frac{\dot{s}}{\ell} = \frac{v}{\ell} = \alpha, \qquad \dot{\alpha} = -\omega_o^2\sin\theta - \frac{1}{\tau}\alpha = f(t,\theta,\alpha)$$

with initial conditions: $at \quad t = 0, \quad \theta = \theta_o, \quad and \quad \dot{\theta} = 0 = \alpha_o$

Which can be solved numerically using the Fourth-Order Runge-Kutta Method:

$$\theta_{i+1} = \theta_i + z\alpha_i + \frac{z}{6}(n_{i1} + n_{i2} + n_{i3})$$

$$\alpha_{i+1} = \alpha_i + \frac{1}{6}(n_{i1} + 2n_{i2} + 2n_{i3} + n_{i4})$$

where:

$$n_{i1} = zf(t_i, \theta_i, \alpha_i)$$
$$n_{i2} = zf(t_i + [z/2], \theta_i + [z/2]\alpha_i, \alpha_i + [n_{i1}/2])$$
$$n_{i3} = zf(t_i + [z/2], \theta_i + [z/2]\alpha_i + [z/4]n_{i1}, \alpha_i + [n_{i2}/2])$$
$$n_{i4} = zf(t_i + z, \theta_i + z\alpha_i + [z/2]n_{i2}, \alpha_i + n_{i3})$$

and

$$z = \Delta t$$

H. CYCLOIDAL PATH [18, 19, 20, 21]

Equations for the cycloid:

Origin, point O, at center of curve, where: $x = 0, y = 0, and\ \beta = 0$.

At the ends of the curve $\beta = \pm\pi/2$.

$$AP = 2a \sin \beta, \qquad DP = 2a \cos \beta,$$
$$y = AB = 2a - DB = 2a - DP \cos \beta = 2a(1 - \cos^2\beta) = 2a \sin^2\beta = a(1 - \cos 2\beta)$$
$$x = H^1D + BP = 2a\beta + a \sin 2\beta = a(2\beta + \sin 2\beta) = 2a(\beta + \sin \beta \cos \beta)$$

Note: AP is tangent to the curve at P and the arc OP is equal to twice the length of the line AP.

$$s = arc\ OP = 2AP = 4a \sin \beta \qquad (H\text{-}1)$$

where a is the radius of the generating circle.

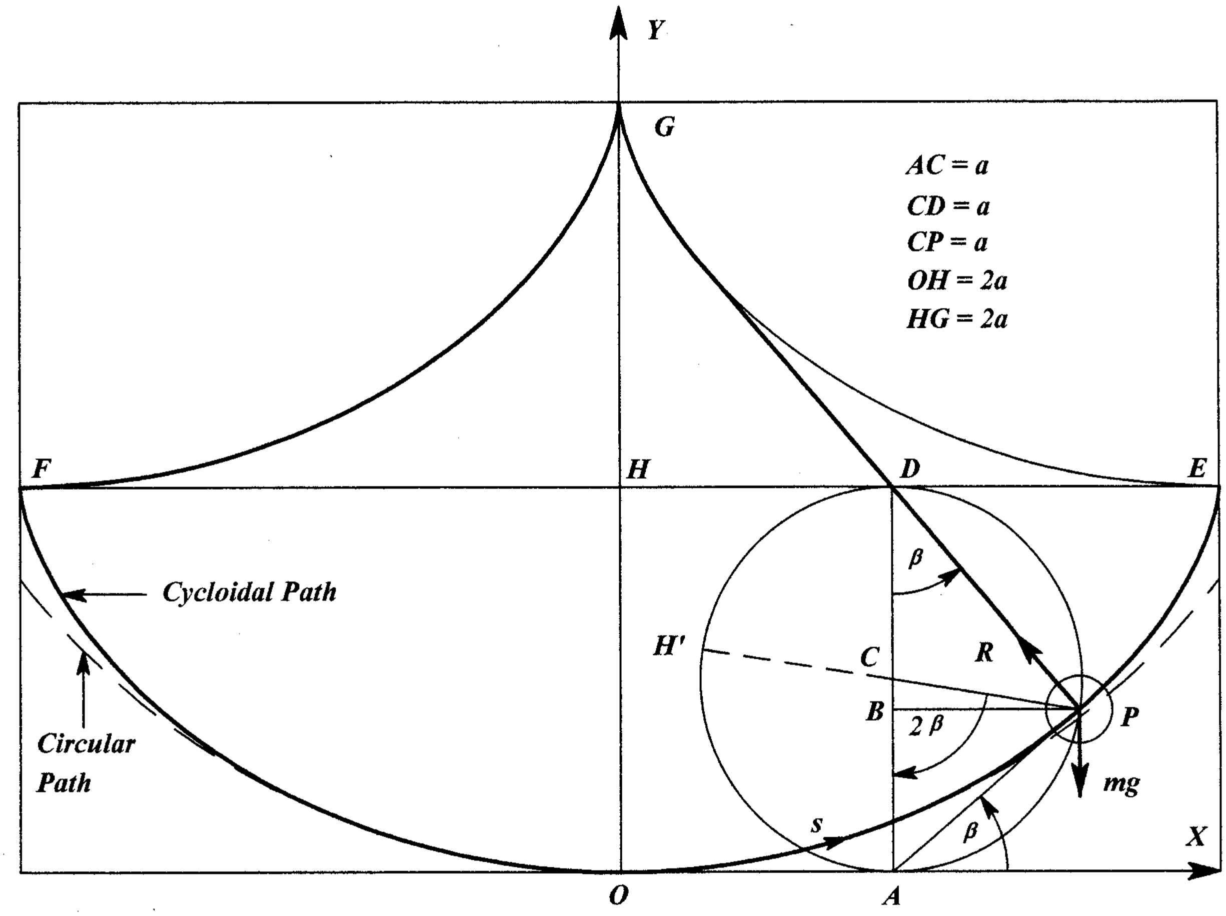

Figure 5.

If R is the reaction normal to the curve, and m the mass of the particle at P, the equations of motion are:

Tangential Force: $$m\frac{d^2s}{dt^2} = -mg\sin\beta \qquad (H\text{-}2)$$

and Normal Force: $$m\frac{v^2}{\rho} = R - mg\cos\beta \qquad (H\text{-}3)$$

Combining Eqs. (H-1) and (H-2) we have:

$$\frac{d^2s}{dt^2} = \frac{-g}{4a}s \qquad (H\text{-}4)$$

This represents simple harmonic motion, and the period is:

$$T_{cy} = 2\pi\sqrt{\frac{4a}{g}} = \frac{1}{f} = \frac{2\pi}{\omega_o}$$

Which shows that the pendulum is isochronous. Integrating Eq. (H-4), we have:

$$v^2 = \left(\frac{ds}{dt}\right)^2 = \frac{-g}{4a}s^2 + c = -4ag\sin^2\beta + c = 4ag(\sin^2\beta_o - \sin^2\beta)$$

if the pendulum started from rest at the point where $\beta = \beta_o$.

We can also derive this using the Principle of Conservation of Energy as in Eq. (B-2):

$$T = \frac{w}{2g}v^2 \qquad and \qquad V = 2aw\sin^2\beta$$

Note that at $y = 2a\sin^2\beta_o$, $v = 0$, therefore:

$$\frac{w}{2g}v^2 + 2aw\sin^2\beta = 2aw\sin^2\beta_o$$

and

$$v^2 = 4ag(\sin^2\beta_o - \sin^2\beta)$$

also from Eq. (H-1) $\rho = \dfrac{ds}{d\beta} = 4a\cos\beta$ = *Radius of Curvature of the Cycloid*

and thus Eq. (H-3) gives:

$$R = mg\cos\beta + mg\frac{(\sin^2\beta_o - \sin^2\beta)}{\cos\beta} = mg\frac{(\cos 2\beta + \sin^2\beta_o)}{\cos\beta}$$

To make the amplitudes of the circular and cycloidal pendulum to be at the same energy level:

For a circular pendulum:	$h = \ell(1 - \cos\theta_o)$	
For a cycloidal pendulum:	$h = 2a\sin^2\beta_o$	Note: $\ell = 4a$
Thus:	$2a\sin^2\beta_o = 4a(1 - \cos\theta_o)$	
and	$\sin^2\beta_o = 2(1 - \cos\theta_o)$	

I. DAMPED CYCLOIDAL PENDULUM [17, 22]

When a particle oscillates along a smooth cycloidal curve in a medium (air) whose resistance is proportional to the velocity, the equations of motion are:

$$m\frac{d^2s}{dt^2} = -mg\sin\beta - 2mk\frac{ds}{dt}, \qquad m\frac{v^2}{\rho} = R - mg\cos\beta$$

From Eq. (H-1) $s = 4a\sin\beta, \quad \rho = \dfrac{ds}{d\beta} = 4a\cos\beta$

and $v = \dfrac{ds}{dt} = 4a\cos\beta\dfrac{d\beta}{dt}, \qquad \omega_o^2 = \dfrac{g}{4a}$

$$2k = \frac{1}{\tau} = \frac{\omega_o}{Q}, \qquad \omega_u = \omega_o\sqrt{1 - \frac{1}{4Q^2}}$$

τ is called the decay time or time constant, then:

$$\frac{d^2s}{dt^2} + \frac{1}{\tau}\frac{ds}{dt} + \omega_o^2 s = 0$$

This is a linear, second-order, differential equation with constant coefficients. For small damping:

$$\frac{1}{2\omega_o\tau} = \frac{1}{2Q} << 1 \quad \text{and } \omega_u \text{ is nearly equal to } \omega_o.$$

We use an exponential to convert a differential equation into an algebraic one:

$$s(t) = A_o e^{-t/2\tau}\cos(\omega_u t + \delta)$$

where A_o is the maximum amplitude.

Now: $$v = \frac{ds}{dt} = \frac{d}{dt}\left[A_o e^{-t/2\tau}\cos(\omega_u t + \delta)\right] = A_o e^{-t/2\tau}\left[\frac{1}{2\tau}\cos(\omega_u t + \delta) + \omega_u \sin(\omega_u t + \delta)\right]$$

Using the initial conditions: at $t = 0$, $v = v_i$, and $s = s_i$

We evaluate the coefficients A_o and δ.

$$s(0) = s_i = A_o \cos\delta$$

$$A_o = s_i/\cos\delta$$

$$v(0) = v_i = A_o[(\omega_o/2Q)\cos\delta + \omega_u \sin\delta]$$

$$v_i = s_i[(\omega_o/2Q) + \omega_u \tan\delta]$$

and

$$\tan\delta = \frac{1}{\sqrt{4Q^2 - 1}}\left[\frac{2Qv_i}{\omega_o s_i} - 1\right]$$

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