VIII. *On the Vibration of a Free Pendulum in an Oval differing little from a Straight Line.*[*]

By G. B. Airy, Esq., Astronomer Royal.

Read May 9, 1851.

In a paper communicated to this Society several years since, and printed in the eleventh volume of their *Memoirs*, I investigated the motion of a pendulum in the case in which it describes an oval differing little from a circle; and I showed that, if the investigation is limited to the first power of ellipticity, and if α is the mean value of the angle made by the pendulum-rod with the vertical, then the proportion of the time occupied in passing from one distant apse to the next distant apse, to the mean time of a revolution, is the proportion of 1 to the square root of $4 - 3\sin^2\alpha$. When α is small, this proportion is nearly the same as the proportion of $\frac{1}{2}$ to $1 - \frac{3}{8}\sin^2\alpha$; or the time of moving from one distant apse to another distant apse is equal to the time of half a revolution divided by $1 - \frac{3}{8}\sin^2\alpha$. This shows that the major axis of the oval is not stationary, but that its line of apses progresses; and that, while the ellipticity is small, the velocity of progress of the apses is sensibly independent of the ellipticity, and may be assigned in finite terms for any value of the mean inclination of the pendulum-rod.

This theorem, however, fails totally when the minor axis of the oval is small. It is then found that the velocity of progress of the apses is nearly proportional to the minor axis. But, although the movement of the pendulum in this case may be defined to any degree of accuracy by infinite series, it does not appear that it can be expressed in finite terms of any ordinary function of the time. This is to be expected, inasmuch as, when the problem is reduced to its utmost state of simplicity by making the minor axis = 0, the motion of the pendulum can be expressed only by series. The utmost, therefore, for which we can hope is, to determine the general form of the curve and the rate of progress of its apses, on the supposition that the minor axis is small, in series proceeding by powers of the major axis. This might be so extended as to include higher powers of the minor axis, if it were judged desirable.

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I have thought that an exhibition of the first steps of solution (carried so far as to include the principal multiplier of the first power of the minor axis) might be acceptable to this Society, not purely as a mechanical problem, but more particularly because it bears upon every astronomical or cosmical experiment in which the movement of a pendulum is concerned. The difficulty of starting a free pendulum, so as to make it vibrate at first in a plane, is extremely great; and every experimenter ought to be prepared to judge how much of the apparent torsion of its plane of vibration is really a progression of apses due to its oval motion.

1. Let a be the length of the pendulum; x, y, z, the co-ordinates of the bob, measured from the point of suspension, z being measured vertically downwards: then we find, without difficulty, the following equations:[1]

$$x\frac{dy}{dt} - y\frac{dx}{dt} - A = 0$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 - 2gz + B = 0,$$

A and B being two constants whose values will depend on the dimensions of the curve described. Eliminating z and its differentials from the second equation, by means of the equation $x^2 + y^2 + z^2 = a^2$ and its differentials, it becomes[2]

$$(a^2 - x^2 - y^2)\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} + \left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)^2 - 2g(a^2 - x^2 - y^2)^{\frac{3}{2}} + B(a^2 - x^2 - y^2) = 0.$$

2. The movement in a stationary oval, whose projection on a horizontal plane does not differ much from an ellipse, will be represented with the utmost generality by the co-ordinates

$$X = b \cdot \cos mt + p,$$
$$Y = c \cdot \sin mt + q,$$

[1]The first equation is the conservation of the z part of the angular momentum at the point of suspension, which is an immediate consequence of the two forces applied to the bob, namely the weight and the tension. Note that only this component is constant, not the whole angular momentum.

The second equation is the conservation of energy, the kinetic energy of the bob being $\frac{1}{2}mv^2$ and its potential energy $-mgz + c$, where c is some constant. (Editor)

[2]The result is obtained by differentiating $x^2+y^2+z^2 = a^2$, which gives $x\frac{dx}{dt}+y\frac{dy}{dt}+z\frac{dz}{dt} = 0$, and by multiplying the conservation of energy equation by $z^2 = a^2 - x^2 - y^2$. (Editor)

where p is a function of t inferior in magnitude to b, and q is a function of t inferior in magnitude to c. And the movement in an oval whose apses' progress will be represented by the ordinates[3]

$$x = X \cdot \cos\psi - Y \cdot \sin\psi$$
$$y = X \cdot \sin\psi + Y \cdot \cos\psi,$$

where ψ is some angle which increases slowly with the time.

3. It appeared to me probable, at first, that the best assumption for the form of ψ would be this: that the area described by the radius vector should be increased, in consequence of the introduction of ψ, in a constant ratio; a condition which is expressed by this equation:

$$\frac{d\psi}{dt} = n \cdot \frac{X\frac{dY}{dt} - Y\frac{dX}{dt}}{X^2 + Y^2}.$$

But, upon proceeding on this assumption to the determination of the form of q, it was found that it would contain a term of the form $E \cdot t \cdot \cos mt$, or $F \cdot t \cdot X$, nearly; a term which was inadmissible, but which indicated clearly that the axis of X must be supposed to revolve with a uniform velocity. The assumption was therefore made

$$x = X \cdot \cos nt - Y \cdot \sin nt$$
$$y = X \cdot \sin nt + Y \cdot \cos nt,$$

n being a small constant: and this assumption has been followed by no difficulties.

4. It is desirable, at as early a stage as possible, to establish the order of the different quantities. The following is the order determined by several tentative steps, which it is not necessary to repeat here: — First, it is considered that $\frac{b}{a}$ and $\frac{c}{b}$ are small quantities of the first order; then it is found that $\frac{p}{a}$ is of the third order, $\frac{q}{a}$ of the fourth order, and $\frac{n}{m}$ of the third order; that g is comparable with m^2a, B with m^2a^2, and A with mbc. It is our object to preserve only so much of the equations as will give the first power of the smallest of these quantities, namely, q; this will be done by means of the first equation, when p has been found; and the second equation may be limited, in the second factor $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$ of its first term, to quantities of the order p, and to corresponding quantities in the other terms.

[3] We have $(x, y) = (X, Y)$ rotated by ψ. (Editor)

5. Giving to x and y their values in terms[4] of X and Y, $x\frac{dy}{dt} - y\frac{dx}{dt} = X\frac{dY}{dt} - Y\frac{dX}{dt} + n(X^2 + Y^2)$. Substituting for X and Y their values, and including all to the order of bq, omitting cq and pq, the first equation becomes[5]

$$mbc - A + nb^2 \cdot \cos^2 mt + mcp \cdot \cos mt - c \cdot \sin mt \cdot \frac{dp}{dt} + mbq \cdot \sin mt + b \cdot \cos mt \cdot \frac{dq}{dt} = 0,$$

an equation which we shall use for determining the values of n and q, when that of p is found.

6. For the second equation: upon differentiating the values of x and y, it will be found that

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{dX}{dt}\right)^2 + \left(\frac{dY}{dt}\right)^2 + 2n\left(X\frac{dY}{dt} - Y\frac{dX}{dt}\right) + n^2(X^2+Y^2).$$

As we wish to preserve only the first power of p or $\frac{dp}{dt}$, which enters here with quantities of the fourth order, it will be found that we have only to retain the terms[6] $m^2b^2 \cdot \sin^2 mt - 2mb \cdot \sin mt \cdot \frac{dp}{dt} + m^2c^2 \cdot \cos^2 mt$. The multiplier $a^2 - x^2 - y^2$ or $a^2 - X^2 - Y^2$ will be reduced to $a^2 - b^2 \cdot \cos^2 mt$. And thus the first part will be reduced to

$$m^2a^2b^2 \cdot \sin^2 mt - 2ma^2b \cdot \sin mt \cdot \frac{dp}{dt} + m^2a^2c^2 \cdot \cos^2 mt - m^2b^4 \cdot \sin^2 mt \cdot \cos^2 mt.$$

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[4] We have

$$x\frac{dy}{dt} - y\frac{dx}{dt} = x\left(\frac{dX}{dt}\sin nt + Xn\cos nt + \frac{dY}{dt}\cos nt - nY\sin nt\right) - y\left(\frac{dX}{dt}\cos nt - Xn\sin nt - \frac{dY}{dt}\sin nt - nY\cos nt\right)$$

and we use $X = x\cos nt + y\sin nt$ and $Y = y\cos nt - x\sin nt$. (Editor)

[5] The previous equation gives:

$$(b\cos mt + p)(cm\cos mt + \frac{dq}{dt}) - (c\sin mt + q)(-bm\sin mt + \frac{dp}{dt}) + n(b^2\cos^2 mt + p^2 + 2pb\cos mt + c^2\sin^2 mt + q^2 + cq\sin mt) = A$$

and the result follows. (Editor)

[6] This follows from $\frac{dX}{dt} = -bm\sin mt + \frac{dp}{dt}$ and $\frac{dY}{dt} = cm\cos mt + \frac{dq}{dt}$. (Editor)

7. $\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)^2 = \left(X\frac{dX}{dt}+Y\frac{dY}{dt}\right)^2 = (-mb^2\cdot\cos mt\cdot\sin mt + \&c.)^2$, of which[7] the only part to be retained is $+m^2b^4\cdot\sin^2 mt\cdot\cos^2 mt$.

8. $-2g(a^2-x^2-y^2)^{\frac{3}{2}} = -2ga^3\left\{1-\frac{3}{2}\cdot\frac{x^2+y^2}{a^2}+\frac{3}{8}\left(\frac{x^2+y^2}{a^2}\right)^2\right\}$, of which[8] there is to be kept

$$\begin{aligned}-2ga^3 + 3gab^2\cdot\cos^2 mt + 6gabp\cdot\cos mt&\\ + 3gac^2\cdot\sin^2 mt&\\ - \frac{3g}{4}\cdot\frac{b^4}{a}\cdot\cos^4 mt.&\end{aligned}$$

9. $+B(a^2-x^2-y^2)$ in like manner is found to be

$$\begin{aligned}+Ba^2 - Bb^2\cdot\cos^2 mt - 2Bbp\cdot\cos mt&\\ - Bc^2\cdot\sin^2 mt.&\end{aligned}$$

10. Collecting now the different parts of the second equation, and arranging them by orders, we have the sum of the following quantities $= 0$:

Principal Terms.

$$-2ga^3 \qquad + Ba^2$$

Terms of the Second Order.

$$m^2a^2b^2\cdot\sin^2 mt + 3gab^2\cdot\cos^2 mt - Bb^2\cdot\cos^2 mt.$$

Terms of the Fourth Order.

$$\begin{aligned}&-2ma^2b\cdot\sin mt\cdot\frac{dp}{dt} + m^2a^2c^2\cdot\cos^2 mt + 6gabp\cdot\cos mt\\ &+ 3gac^2\cdot\sin^2 mt - \frac{3g}{4}\cdot\frac{b^4}{a}\cdot\cos^4 mt - 2Bbp\cdot\cos mt - Bc^2\cdot\sin^2 mt.\end{aligned}$$

This equation is to be solved by successive substitution.

[7] The previous follows because (x,y) is a rotation of (X,Y), and therefore $x^2+y^2 = X^2+Y^2$, and consequently $x\frac{dx}{dt}+y\frac{dy}{dt} = X\frac{dX}{dt}+Y\frac{dY}{dt}$. (Editor)

[8] We have $(1+\varepsilon)^{\frac{3}{2}} = 1+\frac{3}{2}\varepsilon+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2}\varepsilon^2+\cdots = 1+\frac{3}{2}\varepsilon+\frac{3}{8}\varepsilon^2+\ldots$ (Editor)

11. Confining ourselves first to the Principal Terms,[9] $B = +2ga$.

12. Substituting this value in the terms of the second order, and then uniting them to the principal terms, the equation becomes

$$0 = -2ga^3 + Ba^2 + m^2a^2b^2 \cdot \sin^2 mt + 3gab^2 \cdot \cos^2 mt - 2gab^2 \cdot \cos^2 mt,$$

or

$$0 = Ba^2 - 2ga^3 + m^2a^2b^2 + (gab^2 - m^2a^2b^2)\cos^2 mt,$$

whence[10]

$$\begin{aligned} gab^2 - m^2a^2b^2 &= 0, \qquad \text{or} \qquad & m^2 &= \frac{g}{a}, \\ Ba^2 - 2ga^3 + gab^2 &= 0, \qquad \text{or} \qquad & B &= 2ga - g \cdot \frac{b^2}{a}. \end{aligned}$$

13. Substituting the value of m^2 and the first part of B in the terms of the fourth order, and the whole of B in the terms of the second order, and uniting them with the Principal Terms, the equation becomes

$$\begin{aligned} 0 = &- 2ga^3 + Ba^2 + m^2a^2b^2 \cdot \sin^2 mt + 3gab^2 \cdot \cos^2 mt - 2gab^2 \cdot \cos^2 mt \\ &+ g\frac{b^4}{a} \cdot \cos^2 mt - \frac{2gab}{m} \cdot \sin mt \cdot \frac{dp}{dt} + gac^2 \cdot \cos^2 mt \\ &+ 6gabp \cdot \cos mt + 3gac^2 \cdot \sin^2 mt - \frac{3g}{4} \cdot \frac{b^4}{a} \cdot \cos^4 mt \\ &- 4gabp \cdot \cos mt - 2gac^2 \cdot \sin^2 mt, \end{aligned}$$

or

$$\begin{aligned} 0 = &- 2ga^3 + Ba^2 + m^2a^2b^2 + gac^2 \\ &+ \left\{ -m^2a^2b^2 + gab^2 + \frac{gb^4}{a} \right\} \cos^2 mt \\ &- \frac{3g}{4} \cdot \frac{b^4}{a} \cdot \cos^4 mt - \frac{2gab}{m} \cdot \sin mt \cdot \frac{dp}{dt} + 2gabp \cdot \cos mt, \end{aligned}$$

from which the three unknown quantities B, m, and p, are to be obtained.

14. The treatment of this equation requires careful consideration. When an expression, which is to vanish for all values of mt, is arranged by powers of $\cos mt$, it is well known that the coefficient of each power of $\cos mt$ must

[9] The principal terms being equaled to 0, we have $-2ga^3 + Ba^2 = 0$. (Editor)

[10] Compared to the above, we now have a more accurate value of B. (Editor)

separately $= 0$. If we found ourselves at liberty here to unite the terms containing p and $\frac{dp}{dt}$ with the term containing $\cos^4 mt$, and to make their sum $= 0$, the solution would be simple. But upon proceeding with that solution it will be found that the expression obtained for p contains the time t multiplied by a periodic function; a term which is evidently inconsistent with our original assumption that p is always very small compared with b. Some modification must therefore be made in this part of the equation, which at the same time will not prevent us from using the principle of separation by powers of $\cos mt$, to which I have alluded.

15. The modification required is very simple. Arrange the equation in this order:

$$\begin{aligned}0 = &- 2ga^3 + Ba^2 + m^2a^2b^2 + gac^2 \\ &+ \left\{-m^2a^2b^2 + gab^2 + \frac{gb^4}{a} - \frac{9}{8}\cdot\frac{gb^4}{a}\right\}\cos^2 mt \\ &- \frac{3g}{4}\cdot\frac{b^4}{a}\cdot\cos^4 mt + \frac{9}{8}\cdot\frac{gb^4}{a}\cdot\cos^2 mt - \frac{2gab}{m}\cdot\sin mt\cdot\frac{dp}{dt} + 2gabp\cdot\cos mt,\end{aligned}$$

and the last line can now be treated separately without introducing a multiple of the time. Making it $= 0$, we find,[11]

$$p = C\cdot\sin mt - \frac{3}{16}\cdot\frac{b^3}{a^2}\cdot\cos^3 mt.$$

The term $C\cdot\sin mt$ amounts simply to a small alteration in the epoch of time from which the arc in the expression $b\cdot\cos mt$ is measured.[12] Omitting it, we have definitively,

$$p = -\frac{3}{16}\cdot\frac{b^3}{a^2}\cos^3 mt.$$

16. The second line of the equation gives $m^2 = \frac{g}{a}\left(1 - \frac{b^2}{8a^2}\right)$; and the time of describing the oval, or $\frac{2\pi}{m}$, is therefore $2\pi\sqrt{\frac{a}{g}}\cdot\left(1 + \frac{b^2}{16a^2}\right)$ nearly.

17. The first line of the equation may now be used to give a more accurate value of B; but this is of no use or interest, unless we wished to carry the approximation further.

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[11] This can be checked easily by substitution. (Editor)

[12] It also amounts to a small alteration in the amplitude, as can be seen easily if we assume $mt' = mt + \varepsilon$ and compute $b\cos mt + C\sin mt = b'\cos(mt + \varepsilon)$ which leads to $b' = \sqrt{b^2 + C^2}$ and $\varepsilon = \arcsin\left(-\frac{C}{b'}\right)$. (Editor)

18. We shall now revert to the first equation in article 5. Omitting the part $mbc - A$, which merely gives the value of A, we have,

$$nb^2 \cdot \cos^2 mt + mcp \cdot \cos mt - c \cdot \sin mt \cdot \frac{dp}{dt} + mbq \cdot \sin mt + b \cdot \cos mt \cdot \frac{dq}{dt} = 0$$

and, substituting for p the value just found,[13]

$$nb \cdot \cos^2 mt - \frac{3m}{16} \cdot \frac{b^2c}{a^2} \cdot \cos^4 mt - \frac{9m}{16} \cdot \frac{b^2c}{a^2} \cdot \sin^2 mt \cdot \cos^2 mt + mq \cdot \sin mt + \cos mt \cdot \frac{dq}{dt} = 0$$

The solution of this equation gives for q a value containing t multiplied by periodic terms, except we make

$$nb - \frac{3}{8} \cdot \frac{mb^2c}{a^2} = 0,$$

or,

$$\frac{n}{m} = \frac{3}{8} \cdot \frac{bc}{a^2},$$

and then the value of q is found to be,[14]

$$D \cdot \cos mt - \frac{3}{16} \cdot \frac{b^2c}{a^2} \cdot \cos^2 mt \cdot \sin mt$$

[13]The previous equation is divided by b. (Editor)

[14]This can also easily be checked by substitution: using

$$\frac{dq}{dt} = -Dm \sin mt + \frac{3}{8}\frac{b^2cm}{a^2} \cos mt \sin^2 mt - \frac{3}{16}\frac{b^2cm}{a^2} \cos^3 mt,$$

the first equation gives

$$\frac{3}{8}\frac{mb^2c}{a^2}\cos^2 mt - \frac{3m}{16}\frac{b^2c}{a^2}\cos^4 mt - \frac{9m}{16}\frac{b^2c}{a^2}\cos^2 mt + \frac{9m}{16}\frac{b^2c}{a^2}\cos^4 mt + mD \sin mt \cos mt - \frac{3}{16}\frac{b^2cm}{a^2}\sin^2 mt \cos^2 mt - mD \sin mt \cos mt + \frac{3}{8}\frac{b^2cm}{a^2}\cos^2 mt \sin^2 mt - \frac{3}{16}\frac{b^2cm}{a^2}\cos^4 mt = 0$$

Hence

$$(6\cos^2 mt - 3\cos^4 mt - 9\cos^2 mt + 9\cos^4 mt - 3\sin^2 mt \cos^2 mt + 6\cos^2 mt \sin^2 mt - 3\cos^4 mt) \times \frac{mb^2c}{16a^2} = 0$$

or, giving to D the value 0,

$$q = -\frac{3}{16} \cdot \frac{b^2 c}{a^2} \cos^2 mt \cdot \sin mt.$$

19. The result which it was the special object of this investigation to obtain is that given by the equation $\frac{n}{m} = \frac{3}{8} \cdot \frac{bc}{a^2}$. From this we find that the time in which the line of apses would perform a complete revolution is $\frac{8}{3} \cdot \frac{a^2}{bc}$ multiplied by the time of a complete double vibration.

Thus if a pendulum 52 feet long (which performs its double vibration in 8 seconds), vibrated in an ellipse whose major axis is 52 inches and minor axis 6 inches, the line of apses would perform a complete revolution *from this cause* in 30 hours nearly.[15]

If a common seconds' pendulum (which performs its double vibration in 2 seconds) vibrated in an ellipse whose major axis is 4 inches and minor axis $\frac{1}{13}$th-inch, the line of apses would perform a complete revolution *from this cause* in 30 hours nearly.[16]

The direction of rotation of the line of apses is the same as the direction of revolution in the ellipse.

20. It is worthy of remark that the expression $\frac{n}{m} = \frac{3}{8} \cdot \frac{bc}{a^2}$, which has been found on the supposition that c is much smaller than b, will, if we make in it c very nearly equal to b, correspond exactly to the formula cited in the first paragraph of this paper, as found by an accurate investigation when the ellipse approaches very near to a circle. It appears, therefore, very probable, that while b is moderately small, the expression $\frac{n}{m} = \frac{3}{8} \cdot \frac{bc}{a^2}$ is very nearly true for all values of c up to b.

21. Although the principal object of this paper, as mentioned in the beginning, was to point out how far an apparent rotation of the plane of a pendulum's vibration may depend on causes which would exist if the suspension were perfect, and if the point of suspension were unmoved and the

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and therefore

$$-3\cos^2 mt + 3\cos^4 mt + 3\cos^2 mt \sin^2 mt = 0$$
$$-3\cos^2 mt + 3\cos^4 mt + 3\cos^2 mt(1 - \cos^2 mt)) = 0$$

which is obviously true. (Editor)

[15] A complete revolution occurs in $\frac{8}{3}\frac{a^2}{bc} \times 8 = \frac{8}{3}\frac{(52 \cdot 12)^2}{52 \cdot 6} \times 8$ seconds, hence in 29.58 hours. (Editor)

[16] A second's pendulum is 16 times shorter than a pendulum performing its double vibration in 8 seconds, hence the revolution time in the previous example ought to be multiplied by $\frac{1}{16^2} \times \frac{52}{4} \times \frac{6}{\frac{1}{13}} \times \frac{1}{4} = \frac{13^2 \cdot 3}{512} \approx 0.99$. The revolution of the line of apses is again performed in about 29.3 hours. (Editor)

direction of gravity invariable; still it may not be uninteresting to point out how an effect in some respects similar may be produced by a fault in the suspension. If a pendulum be suspended by a wire passing through a hole in a solid plate of metal, the orifice of that hole may be oval. If the wire be part of a thicker rod tapering to the size of the wire, it may taper unequally on different sides. In either case there will be two planes of vibration, at right angles to each other, in which if the pendulum is vibrating, it will continue to vibrate, and in one of which the time of vibration is greater, and in the other less, than in any other plane. And, the amplitude of vibration being very small, the complete motion may be found by compounding the vibration corresponding to these two planes.

22. Let x and y be the horizontal co-ordinates of the bob in the two principal planes; and let the two vibrations be represented by $x = g \cdot \cos kt$, $y = h \cdot \cos lt$, where l differs very little from k. While t is moderately small, the proportion $\frac{y}{x}$ will not sensibly differ from $\frac{h}{g}$, which denotes that the pendulum will at first apparently vibrate in a plane, making with the plane of xz an angle[17] α, whose tangent is $\frac{h}{g}$. But after a long time T, the place of the bob at $T + t'$ will be defined by the co-ordinates $x = g \cdot \cos(kT + kt') = (g \cdot \cos kT) \cos kt' - (g \cdot \sin kT) \sin kt'$, $y = h \cdot \cos(lT + lt') = (h \cdot \cos lT) \cos lt' - (h \cdot \sin lT) \sin lt'$. If we now refer the motion to the two co-ordinates r, s, inclined at the angle θ to x and y, we shall have,

$$
\begin{aligned}
r &= x \cdot \cos\theta + y \cdot \sin\theta \\
s &= -x \cdot \sin\theta + y \cdot \cos\theta
\end{aligned}
$$

and remarking, that for a small time $\cos kt'$ is sensibly equal to $\cos lt'$ and $\sin kt'$ is sensibly equal to $\sin lt'$,

$$
\begin{aligned}
r = (g \cdot \cos kT \cdot \cos\theta + h \cdot \cos lT \cdot \sin\theta) \cos kt' \\
- (g \cdot \sin kT \cdot \cos\theta + h \cdot \sin lT \cdot \sin\theta) \sin kt'
\end{aligned}
$$

$$
\begin{aligned}
s = (-g \cdot \cos kT \cdot \sin\theta + h \cdot \cos lT \cdot \cos\theta) \cos kt' \\
+ (g \cdot \sin kT \cdot \sin\theta - h \cdot \sin lT \cdot \cos\theta) \sin kt'
\end{aligned}
$$

Let θ be so determined that

$$
\frac{g \cdot \sin kT \cdot \cos\theta + h \cdot \sin lT \cdot \sin\theta}{g \cdot \cos kT \cdot \cos\theta + h \cdot \cos lT \cdot \sin\theta} = \frac{-g \cdot \cos kT \cdot \sin\theta + h \cdot \cos lT \cdot \cos\theta}{g \cdot \sin kT \cdot \sin\theta - h \cdot \sin lT \cdot \cos\theta}
$$

[17]Note that this angle α is not the same as the angle between the rod and the vertical. (Editor)

which gives[18] $\tan 2\theta = \frac{2gh}{g^2-h^2}\cos(kT-lT) = \tan 2\alpha\cdot\cos(kT-lT)$; and let either of these fractions $= \tan\psi$. Then, while t' is moderately small,[19]

$$r = \sqrt{g^2\cdot\cos^2\theta + h^2\cdot\sin^2\theta + 2gh\cdot\sin\theta\cdot\cos\theta\cdot\cos(kT-lT)}\,\cos(kt'+\psi)$$

$$s = \sqrt{g^2\cdot\sin^2\theta + h^2\cdot\cos^2\theta - 2gh\cdot\sin\theta\cdot\cos\theta\cdot\cos(kT-lT)}\,\sin(kt'+\psi)$$

from which it is evident that the pendulum now describes an ellipse, whose major axis coincides with the axis of r. The position of this major axis, which is defined by the equation $\tan 2\theta = \tan 2\alpha\cdot\cos(kT-lT)$, is constantly varying, except in the particular cases when α is zero or a multiple of $45°$.

23. The measure of the variation of position of the major axis will be obtained from the expression for[20] $\tan\overline{2\alpha-2\theta} = \frac{\sin 4\alpha\cdot\sin^2\frac{1}{2}(kT-lT)}{\cos^2\frac{1}{2}(kT-lT)+\cos 4\alpha\cdot\sin^2\frac{1}{2}(kT-lT)}$

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[18]The result follows easily from the previous equation, and then uses $\tan\alpha = \frac{h}{g}$ and $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$. (Editor)

[19]The result follows from the previous expressions for r and s, and these expressions were only valid for small values of t'. The derivation is as follows. First, if we assume that $r = p\cos(kt'+\psi')$ and $s = q\sin(kt'+\psi')$ for some ψ', the identification of $p\cos\psi'\cos kt' - p\sin\psi'\sin kt'$ with the above expression for r leads to

$$\begin{aligned} p^2 &= (p\cos\psi')^2 + (p\sin\psi')^2 \\ &= g^2\cos^2\theta\cos^2 kT + h^2\sin^2\theta\cos^2 lT + 2gh\sin\theta\cos\theta\cos kT\cos lT \\ &+ g^2\cos^2\theta\sin^2 kT + h^2\sin^2\theta\sin^2 lT + 2gh\sin\theta\cos\theta\sin kT\sin lT \\ &= g^2\cos^2\theta + h^2\sin^2\theta + 2gh\sin\theta\cos\theta\cos(kT-lT) \end{aligned}$$

Then, given this identification, ψ' is necessarily such that $\tan\psi'$ is defined as the first fraction above, hence $\psi' = \psi$. We therefore conclude the given new expression for r.

Similarly, we obtain

$$q^2 = g^2\sin^2\theta + h^2\cos^2\theta - 2gh\sin\theta\cos\theta\cos(kT-lT)$$

and the new expression for s. (Editor)

[20]Using $\tan(a+b) = \frac{\tan a+\tan b}{1-\tan a\tan b}$, $\cos 2a = 1-2\sin^2 a$ and $\tan 2\alpha = \frac{\sin 4\alpha}{2\cos^2 2\alpha}$, we have

$$\begin{aligned} \tan(2\alpha-2\theta) &= \frac{\tan 2\alpha - \tan 2\theta}{1+\tan 2\alpha\tan 2\theta} = \frac{\tan 2\alpha\cdot(1-\cos(kT-lT))}{1+\tan^2 2\alpha\cdot\cos(kT-lT)} \\ &= \frac{\tan 2\alpha\cdot 2\sin^2\frac{1}{2}(kT-lT)}{1+\tan^2 2\alpha\cdot(1-2\sin^2\frac{1}{2}(kT-lT))} = \frac{\sin 4\alpha\cdot\sin^2\frac{1}{2}(kT-lT)}{\cos^2 2\alpha+\sin^2 2\alpha\cdot(1-2\sin^2\frac{1}{2}(kT-lT))} \\ &= \frac{\sin 4\alpha\cdot\sin^2\frac{1}{2}(kT-lT)}{1-2\sin^2 2\alpha\cdot\sin^2\frac{1}{2}(kT-lT)} = \frac{\sin 4\alpha\cdot\sin^2\frac{1}{2}(kT-lT)}{\cos^2\frac{1}{2}(kT-lT)+(1-2\sin^2 2\alpha)\sin^2\frac{1}{2}(kT-lT)} \\ &= \frac{\sin 4\alpha\cdot\sin^2\frac{1}{2}(kT-lT)}{\cos^2\frac{1}{2}(kT-lT)+\cos 4\alpha\cdot\sin^2\frac{1}{2}(kT-lT)} \end{aligned}$$

(Editor)

and while $kT - lT$ is not very great, this may be expressed nearly enough by $\sin 4\alpha \cdot \tan^2 \frac{1}{2}(kT - lT)$. If we take the value of the same expression for α', where $\alpha' = \alpha - 45°$, we find a value equal in magnitude but opposite in sign. It appears, therefore, that the effect of faulty suspension may be sensibly eliminated between two experiments in which the azimuths of the first vibration differ by 45°: and it may be prudent, in making any important experiment, thus to change the commencement-azimuths in successive trials.

G. B. AIRY

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