XVIII. *On the Forms of the Teeth of Wheels.*[*]

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[Read *May 2, 1825.*]

The investigation of the forms proper for the teeth of wheels is a useful and interesting inquiry. The mechanical principles are very simple, and the geometrical propositions on which it is immediately made to depend, admit of being put in an elegant form. But all the theories which have yet been given, are, I believe, very imperfect. Euler in the New Petersburgh Commentaries [1] for 1760 has treated the subject with great generality; but the analytical method which he has used is very unfavourable for the discovery of the most obvious properties of the curves. In all the other theories that I have seen, no forms are mentioned but the involute of a circle, and the epicycloid and hypocycloid. In this paper I propose to consider generally the figures which must be given to the teeth of wheels to insure uniformity of action. The curves above alluded to, though probably the most convenient of all, I shall shew are particular cases of a very general construction: and the demonstration which has usually been given for them, I shall apply to every other case.

That the mechanical effect which one wheel produces upon another, may in all positions be the same, it is necessary that the line perpendicular to the surfaces of the teeth,[2] at the point of contact, intersect the line joining the centers at a fixed point, which divides that line into two parts, the ratio of which is the mechanical power. When this holds, the proportion of the angular velocities will be constant. For let A and B (Plate XV. Fig. 1.) be the centers of the wheels, C the point through which the line of action passes: D the point of contact: upon moving the wheels with the teeth still

[*]Transactions of the Cambridge Philosophical Society, vol. 2, 1827, p. 277–286 (and plate XV). Typeset and annotated by Denis Roegel, 26 October–14 November 2006. Version of December 3rd, 2006.

[1] *Novi commentarii academiæ scientiarium Petropolitanæ* **5** (1754/5), 1760, p. 299-316. (Editor)

[2] This is called the line of action. (Editor)

in contact through a very small angle, D in one tooth [3] will be carried to F, and in the other to G, FG being ultimately parallel to the tangent at D, or perpendicular to CD, and DF, DG, perpendicular to AD, BD respectively. Then,[4]

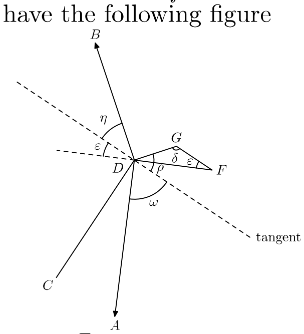
$$FD : GD :: \sin G : \sin F :: \sin BDC : \sin ADC :: \frac{BC}{BD} : \frac{AC}{AD};$$

therefore the angular velocities, which are as $\frac{FD}{AD} : \frac{GD}{BD}$, will be as $BC : AC$, a constant ratio.[5] If then with centers A and B circles be described passing through C, and these circles revolve so as to make the velocities of their circumferences equal, the teeth of the wheels, if properly formed, will be in contact, and the normals to both will pass through C. These circles we shall call the principal circles of the wheels.

If the normals from every point of the tooth should be equally inclined to the tangents of the circle at the points where they meet the circle, they

[3] We consider actually two points, D_1 on A, and D_2 on B. D_1 is carried to F (on a circle centered on A) and D_2 is carried to G (on a circle centered on B). (Editor)

[4] $a : b :: c : d :: e : f ...$ means $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \cdots$. Now, since FG is parallel to the tangent to the teeth at D, we have the following figure



and $\delta = \pi - \rho = \frac{\pi}{2} + \eta = \widehat{BDC}$, $\varepsilon = \frac{\pi}{2} - \omega = \widehat{ADC}$.

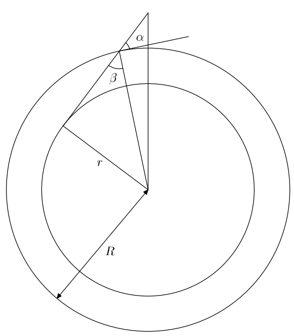
We have $\frac{\sin \widehat{FGD}}{FD} = \frac{\sin \widehat{DFG}}{GD}$, hence $\frac{FD}{GD} = \frac{\sin \widehat{FGD}}{\sin \widehat{DFG}} = \frac{\sin \delta}{\sin \varepsilon} = \frac{\sin G}{\sin F} = \frac{\sin \widehat{BDC}}{\sin \widehat{ADC}}$. On the other hand, we have also $\frac{\sin \widehat{BDC}}{BC} = \frac{\sin \widehat{BCD}}{BD} = \frac{\sin \widehat{ACD}}{BD}$ and $\frac{\sin \widehat{ACD}}{AD} = \frac{\sin \widehat{ADC}}{AC}$. Therefore $\frac{\sin \widehat{BDC}}{BC} = \frac{AD}{AC \cdot BD} \sin \widehat{ADC}$ and $\frac{\sin \widehat{BDC}}{\sin \widehat{ADC}} = \frac{\left(\frac{BC}{BD}\right)}{\left(\frac{AC}{AD}\right)}$ which completes the proof. (Editor)

[5] $\frac{FD}{AD}$ is the angle by which wheel A has turned, when FD is very small. $\frac{GD}{BD}$ is the angle by which wheel B has turned, when GD is very small. Hence $\frac{\left(\frac{FD}{AD}\right)}{\left(\frac{GD}{BD}\right)} = \frac{FD}{GD} \times \frac{BD}{AD} = \frac{BC \cdot AD}{BD \cdot AC} \times \frac{BD}{AD} = \frac{BC}{AC}$. (Editor)

evidently would if produced be tangents to a circle, whose radius : radius of circle described :: cosine of inclination of normal with tangent of circle described : 1.[6] In this case both teeth would be involutes of circles. If the inclinations are not equal, we must make use of the following theorem. It is always possible to find a curve which by revolving upon a given curve, shall by some describing point, in the manner of a trochoid, generate a second given curve: provided that the normals from all points of the second curve meet the first.

To prove this let AB, (Fig. 2.) be the first curve, AC the second; from the points C and E, which are very near, draw the normals CD, EF; if a describing point P be taken, and PQ, PR, be made respectively equal to CD, EF, and QR equal to DF, and this process be continued, a curve will be formed, which by revolving upon BA, will, by the describing point P, generate the curve AC.[7] For if Q coincide with D, then R will afterwards coincide with F, and so on for all succeeding points, since $QR = DF$. Also $DC = QP$, &c. And the angles made by these with the tangents are equal. For the cosines of the angles, drawing DG, QS, perpendicular to EF, PR, are $\frac{FG}{FD}$ and $\frac{RS}{RQ}$, in which the numerators are the differences of equal lines,[8] and the denominators are equal. Hence P will describe AC. And the formation of the curve RQ is always possible, because RQ is greater than RS; for FD is necessarily greater than FG. As an example of this, suppose it were required to find the curve, which revolving on one straight line AB, (Fig. 3.) would generate another straight line AC. Since the angles made by the line PQ with the tangent, must be constant, it follows, that the curve would be the logarithmic spiral, P being its pole.[9]

[6] We have the following construction, where $\cos\alpha = \sin\beta = \frac{r}{R}$. Hence $r : R :: \cos\alpha : 1$.



(Editor)

[7] P is a fixed point on a new curve. (Editor)

[8] The reasoning uses the fact that $EG = CD$, which is true because EG and CD are normals to AC, and because E and C are very near. (Editor)

[9] A logarithmic spiral is a curve that makes a constant angle with its radius vector. We can obtain its polar equation as follows. Let the polar equation of the curve be $\rho = r(\theta)$.

The entire theory of the teeth of wheels, may now be included in this proposition. If the tooth HD, (Fig. 4.) be generated by the revolution of any curve on the outside of the circle HC, and if DK be generated by the revolution of the same curve in the same direction, in the inside of the circle KC, then the normal at the point of contact of the teeth, will pass through C. For let the generating curve be brought to the position LC, so as to touch the circle HC at C; DC will be the normal of HD at D; and that the teeth may be in contact, the same generating curve in the other circle must touch KC at C; in which case it will coincide with this; D therefore will be in the surfaces of both of the teeth, and CD the normal of both at that point; therefore they will touch at D, and the line of action CD, will pass through the fixed point C.[10] If now we give equal velocities [11] to the circumferences CH, CK, the same will be found at all times to be true. These forms then are proper for the teeth of wheels.

Suppose then this problem proposed. Given the form of the teeth of one wheel, to find the form of those of another, that they may work together correctly.[12] The following is the obvious solution. Divide the line joining the centers of the circles at C, into two parts, whose proportion is the mechanical power. Describe the circles CH, CK. Find the curve which by revolving upon CH, will generate the given tooth HD. Make the same curve revolve in CK, and with the same describing point let it generate KD; KD is the form required.

The usual construction of the involute of a circle, would seem to require

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Then the radius vector of the curve is directed along $\vec{r} = (r\cos\theta, r\sin\theta)$ and its tangent along $\vec{v} = (\frac{dr}{d\theta}\cos\theta - r\sin\theta, \frac{dr}{d\theta}\sin\theta + r\sin\theta)$. Then $\vec{r}\cdot\vec{v} = r\frac{dr}{d\theta}\cos^2\theta - r^2\sin\theta\cos\theta + r\frac{dr}{d\theta}\sin^2\theta + r^2\sin\theta\cos\theta = r\frac{dr}{d\theta} = ||\vec{r}||\cdot||\vec{v}||\cdot\cos\alpha$. Hence $\cos\alpha = \frac{r\frac{dr}{d\theta}}{r\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}} = \frac{\frac{dr}{d\theta}}{\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}}$. If this expression is constant, it follows easily that $\frac{dr}{d\theta} = ar$, and therefore $\frac{dr}{r} = ad\theta$, hence $\ln r = a\theta + c$ and $r = be^{a\theta}$, where a and b (and c) are constants. (Editor)

[10] This can be summarized as follows. Take a point D on one of the generated curves. Find the pair XP of the generating curve such that $CD = XP$ and draw the generating curve with X at C. The P coincides with D. Likewise, the corresponding point D' on the mating tooth is found such that $BD' = BD$. The generating curve is positioned in the same way. (It is the same curve.) Then, since the generating curve rolls on HC or KC, C is the center of instantaneous rotation, and therefore CD is perpendicular to the tangent of both teeth. (Editor)

[11] Airy probably meant that the circumferences receive some uniform motion, but the velocities of both circumferences can be different. (Editor)

[12] By *correctly*, it is meant that the ratio of velocities will be constant. (Editor)

that the circles AH, and BK, should be separated.[13] If however DH be the involute formed in the usual way from the circle MN, (Fig. 5.) the normal CM will be inclined at a constant angle to CA, (since its sine $= \frac{AM}{AC}$), and the construction given before shews [14] that the involute HD may be generated by the revolution of a logarithmic spiral upon CH, the describing point being the pole of the spiral, and the angle between its radius and tangent, the same as the angle made by MC, with the tangent of the circle at C. In the same way the revolution of this spiral in the second circle will generate another involute; and hence if the teeth of one wheel be involutes, those of the other wheel must also be involutes. The generating circles of the involutes must have radii proportional to AC, BC.[15]

It will be seen immediately, that we may if we please suppose successive parts of the curve described by different generating curves; or we may make one curve revolve on the outside of the circle CH, and another on the inside, making the same curves revolve on the inside and outside of CK respectively, and thus an infinite variety of curves may be found. The construction last mentioned gives forms approximating most nearly to the usual forms of teeth. We may even give different forms to different teeth but this probably would not be desirable.

It may be desirable to know when the nature of the teeth will admit of an alteration in the distance of the centers of the wheels. Suppose then DL and FP, (Fig. 6.) to be the principal circles when the wheels are in the first position;[16] KS and HR, the principal circles when the distance of the centers is increased. Suppose in the first position C was in contact with E, and M with O; suppose in the second position, G and Q are in contact with E and O; draw normals to all these points as in the figure. Since the wheels in the first position work correctly, by supposition, the angles at D and N will equal those at F and P. And if they work correctly in the second position, HG will $= KE$, &c. HR will $= KS$, and the angles at H and R will equal those at K and S. By attending to this condition, when the tooth EO is given, we

[13] This is because we usually take a circle and add the involutes beyond the circle. This wouldn't be possible with the principal circles. (Editor)

[14] Fig. 3 showed the rolling of a logarithmic spiral on a straight line, producing a straight line, but the same spiral rolling on other curves generates other curves. On a circle, it generates an involute of a smaller circle. (Editor)

[15] This is a consequence of the relationship between the angles of the normals with the principal circles which must be the same. Then $\frac{AM}{AC} = \frac{BN}{BC}$ and hence $\frac{AM}{BN} = \frac{AC}{BC}$. (Editor)

[16] In that position, these two circles are tangent, but they are shown apart. The position of the principal circles are determined by the velocity ratio. (Editor)

can always form a tooth CQ, which will work with it in two positions of the wheels. Since the angles at H and R equal those at K and S, the angles at L and T will equal those at F and P;[17] and therefore will equal those at D and N. It is evident that this condition will always be satisfied, if CQ be the involute, and therefore if the teeth be involutes, the distance of the centers may be altered, to any degree, allowing the teeth to act on each other.

In all, however, that has yet been stated, we have only considered the mathematical conditions of the contact of two curves. That these forms may be applicable in practice, it is necessary that the curvature of the convexity of one tooth, should be greater than that of the concavity of the other, or else that both should be convex. For this purpose we must investigate the curvature at any point.

Take then two points on the circle near each other, and the two points of the generating curve which will touch them; join these with the center of curvature of the generating curve, and with the describing point; let ϕ, θ, ψ, (Fig. 7.) be the small angles at the center of curvature, the describing point, and the center of the circle;[18] suppose the lines from the describing point, when in contact with the circle, to be produced [19] respectively, and let the angle at their point of intersection $= \chi$. Also let α and β be the angles which those lines make with the radii of the circle. Then we shall have [20]

$$\theta - \phi = \alpha - \beta; \quad \psi - \chi = \alpha - \beta; \quad \therefore \chi = \psi + \phi - \theta.$$

But calling R the radius of the circle, r the radius of curvature,[21] s the distance of the describing point, x the distance of the point of intersection,[22]

$$\psi = \frac{\text{arc}}{R}; \quad \phi = \frac{\text{arc}}{r}; \quad \theta = \frac{\text{arc.}\cos\alpha}{s}; \quad \chi = \frac{\text{arc.}\cos\alpha}{x};$$

$$\therefore \frac{\cos\alpha}{x} = \frac{1}{R} + \frac{1}{r} - \frac{\cos\alpha}{s}; \quad x = \frac{\cos\alpha}{\frac{1}{R} + \frac{1}{r} - \frac{\cos\alpha}{s}};$$

$$\therefore x + s = s\frac{\frac{1}{R} + \frac{1}{r}}{\frac{1}{R} + \frac{1}{r} - \frac{\cos\alpha}{s}} = \text{rad. of curvature of tooth};$$

$$\therefore \text{curvature} = \frac{1}{s} \cdot \frac{\frac{1}{R} + \frac{1}{r} - \frac{\cos\alpha}{s}}{\frac{1}{R} + \frac{1}{r}} = \frac{1}{s} - \frac{\frac{\cos\alpha}{s^2}}{\frac{1}{R} + \frac{1}{r}}.$$

[17] This follows from simple proportions. (Editor)

[18] In Fig. 7, the generating curve is drawn above the principal circle. (Editor)

[19] = prolonged. (Editor)

[20] By consideration of triangles, we have obviously $\alpha + \phi = \beta + \theta$ and $\alpha + \chi = \beta + \psi$. (Editor)

[21] of the describing curve. (Editor)

[22] In the expressions for θ and χ, arc.$\cos\alpha$ is arc $\times \cos\alpha$, not the arccos function. All the equations follow easily. (Editor)

From an examination of this expression, it appears, that when α is $< 90^\circ$, r may be positive or negative, but must be less than the radius of the circle in the same direction; when α is $> 90^\circ$, r may be positive or negative, and must be greater than the radius in the same direction.

If then, as is the case in general, α be $< 90^\circ$, that part of the tooth which is without the circle,[23] must be formed by the revolution of some curve upon the circle, and that which is within it by the revolution of some curve within the circle. This kind of tooth is represented in Fig. 4. But if α may be $> 90^\circ$, the whole of the teeth may be formed by the revolution of a single curve; an instance of this is represented in (Fig. 8.) where the teeth GH and KL are formed by the motion of MN, carrying the describing point P. In the last case, if the curve be a circle equal to one of the circles, one tooth will be reduced to a point, the other will be an epicycloid or epitrochoid, according as the describing point is in the circumference of the circle, or in any other part.

It will easily be seen, that where the acting surface of the driving tooth is above the circle, the action takes place after passing the line joining the centers; when below the circle, it is before passing that line. Now practical men always think it proper, that the action should take place only after passing the line of centers. It is thought necessary that the direction of the friction should be such as to wipe off the dust, &c. from the teeth. For this purpose then, the curve which has been found for the lower part of the teeth, must be considered as a limit which that tooth must not reach. In the case in which the whole is formed by the revolution of one curve, the whole action takes place after passing the line of centers.

To find what the friction really amounts to, we have merely to observe, that in Fig. 1. if D be brought to G in one tooth, and to F in the other, GF is the friction, and if $BDC = \alpha$, $FG : FD :: \sin ADB : \sin\alpha$;[24] therefore frictional motion $\propto \dfrac{\sin ADB}{\sin\alpha} \propto \dfrac{\sin ADB}{\sin BCD}$ nearly,[25] (the teeth being so small, that DF may be considered as nearly representing the motion of the circumference.) Also the pressure occasioned by a given force in given circumstances $\propto \dfrac{1}{\sin BCD}$; and the mechanical effect of friction is proportional to the pressure by which it is caused multiplied by the velocity of the rubbing

[23] that is, beyond the principal circle. (Editor)

[24] With the notations of note 4, we have $\dfrac{FG}{\sin(\delta+\varepsilon)} = \dfrac{FD}{\sin\delta}$, that is $\dfrac{FG}{\sin\widehat{ADB}} = \dfrac{FD}{\sin\widehat{BDC}}$, QED. (Editor)

[25] We approximate $\sin\widehat{BDC}$ by $\sin\widehat{BCD}$, which is valid when the teeth are small (in proportion). (Editor)

surfaces; and therefore $\propto \dfrac{\sin ADB}{\sin^2 BCD}$ nearly. The numerator is proportional to the distance from the line of centers; and therefore will be the same for all teeth, when that distance is the same. But the denominator is largest when the face of the tooth is parallel to the radius of the circle. I imagine then that it is advisable to make the teeth work a little before as well as a little after the line of centers. And I should think that a tooth similar to that formed by the union of the epicycloid and hypocycloid, is preferable to any other form whatever. For the line of action is always very nearly perpendicular to the radius; by which means not only is the friction made much less, but also the strain upon the axes is considerably diminished.

If it be thought desirable to prevent back-lashing, this can be done by giving proper forms on the same principles to the faces of the teeth, which are not the working faces. But the chance of very greatly increasing the friction, makes the propriety of this consideration very doubtful.

The whole of what has been stated with regard to circles, it is evident will apply equally to straight lines. Thus the teeth of rack-work may be formed as a combination of cycloids, in which case those of the wheel must consist of epicycloids, and hypocycloids; they may be straight, which will make those of the wheel the involutes of a circle, (both being generated by the revolution of a logarithmic spiral;) they may be mere pins, in which case the teeth of the wheel will be involutes, or curves described in nearly the same manner as involutes. In this case, and in the case of trundles, if it be required to take account of the diameter of the pins, this will be done by taking a curve, whose normal distance from the curve found by considering them as points, shall at all parts be equal to the radius of the pin. Or the form of the teeth may be found by the general theorem.

For crown wheels, as the contrate wheel of a watch, the teeth without sensible error may have the same form as for rackwork. The theory may be extended to bevelled wheels, without any difficulty.

There is one case which ought to be mentioned particularly. It may be desired that the teeth of one wheel have plane surfaces passing through the axis of the wheel. Since a straight line is the hypocycloid, in which the radius of the generating circle is half that of the fixed circle, the teeth of the other wheel must be epicycloids, the radius of the generating circle being half that of the first wheel. The action here takes place entirely after the line of centers, and the direction of the action is nearly perpendicular to that line. I imagine this to be a good construction for pinions with a small number of teeth driven by a large wheel. If each tooth consist of a line within the principal circle, and an epicycloid without it, the radius of the generating circle of each epicycloid, being half that of the other principal circle, a very good form will be produced. The action takes place before as well as after the line of centers, and is always nearly perpendicular to that line. The figure usually given to the teeth of watch-wheels approaches very nearly to this.

I have confined my attention entirely to uniformity of action, and uniformity of motion, as I conceive them to be of far greater consequences than the diminution of friction. The friction can never be made $= 0$, except the point of contact be always in the line of centers; a condition which may be satisfied by an infinite number of curves, and amongst others by two logarithmic spirals. But the mechanical action and the motion would be dreadfully irregular.

I am informed by engineers, that this question is now little more than one of mere curiosity. In consequence of the very extensive use of iron, where wood was formerly employed, the teeth of wheels are now made so small, that it is of little consequence whether they have, or have not, the exact theoretical form. Almost all teeth are now made with plane faces passing through the axis of the wheel, and are expected to wear themselves in a short time into proper forms. This is the case with nearly all the modern iron wheels that I have examined; in the wheels of clock and watch-work, some attention to the figure is however thought necessary.

G. B. AIRY

Trinity College,
April 30, 1825.

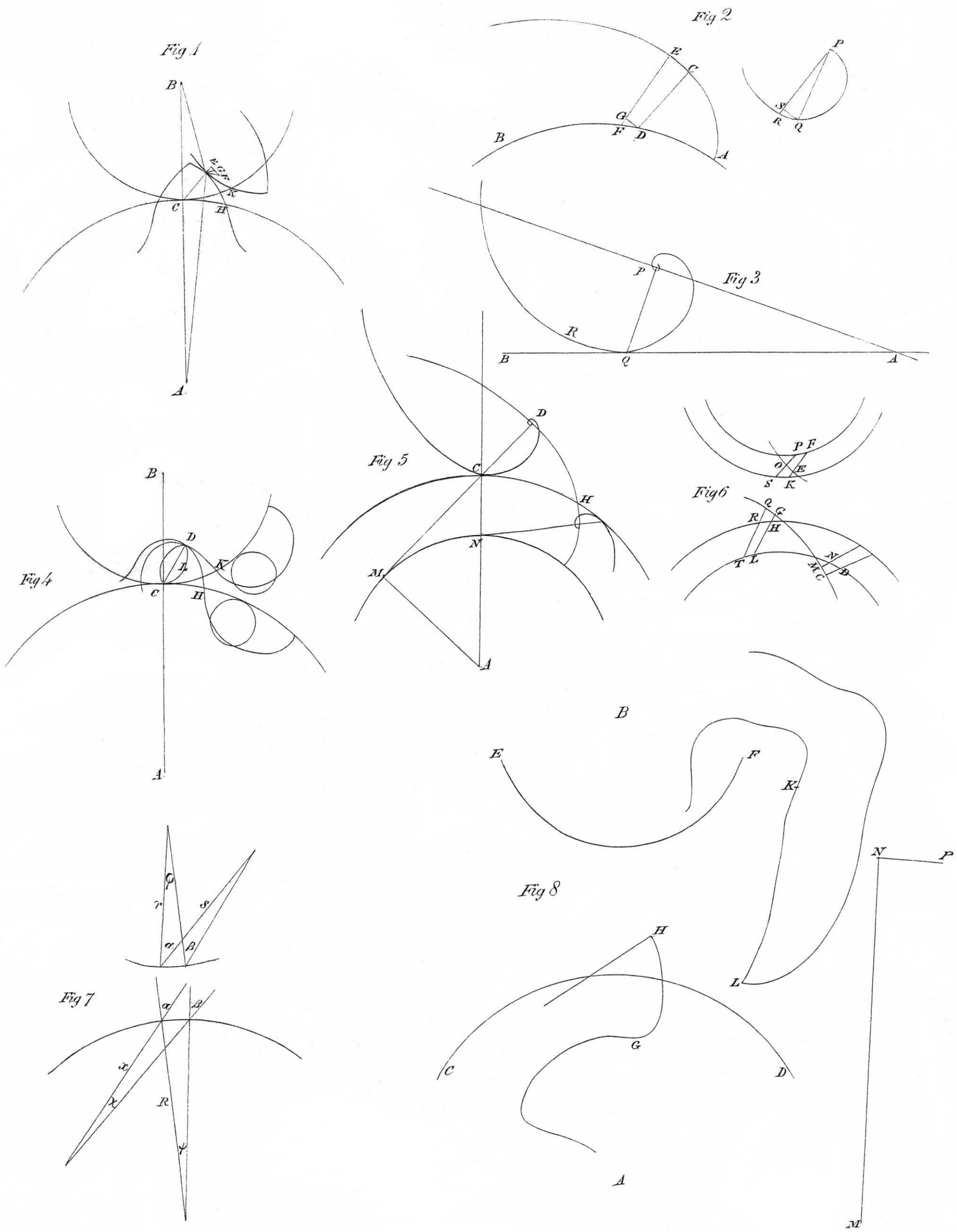
Fig 1
Fig 2
Fig 3
Fig 4
Fig 5
Fig 6
Fig 7
Fig 8